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APPLICATIONS OF NONSTANDARD ANALYSIS TO IDEAL BOUNDARIES IN POTENTIAL THEORY

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ABSTRACT
A solution is given of the generalized Dirichlet problem for an arbitrary compactification of a Brelot harmonic space. A method of obtaining the Martin–Choquet integral representation of positive harmonic functions is given, and the existence is established of an ideal boundary Δ supporting the maximal representing measures for positive bounded and quasibounded harmonic functions with almost all points of Δ being regular for the Dirichlet problem.

1. Introduction

In this paper we shall use standard methods of potential theory and Abraham Robinson's nonstandard analysis [33] to extend potential theoretic properties of the unit disc to more general domains. In particular, we shall establish the existence of an ideal boundary Δ for a general domain that is similar to the boundary of R. S. Martin [29] in terms of representing bounded and quasibounded harmonic functions. The boundary Δ, however, has the property that almost all points (with respect to harmonic measure) are regular for the Dirichlet problem. M. G. Shur [35] has shown that the Martin boundary does not, in general, have this property.

The results of this paper will be established in the setting of M. Brelot's axiomatic potential theory [4], [5]. In addition, we assume that 1 is superharmonic. Examples of this axiomatic setting are given by harmonic functions and indeed by the C^2-solutions u of an elliptic differential equation of the form

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$$\sum a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum b_i \frac{\partial u}{\partial x_i} + cu = 0$$

on a region in Euclidean space where $\sum a_{ik} x_i x_k$ is a positive definite quadratic form, the coefficients of the equation satisfy a local Lipschitz condition, and $c \leqq 0$. (See [18,chap. VII], and [12, p. 326].) Brelot's axioms are also satisfied by the solutions of $\Delta u = Pu$ on an open Riemann surface W, where P is a smooth nonnegative density on W (see [22]). Any result established here is established for each of these special cases with no further verification being necessary. The reader who is interested in only one of the cases subsumed by Brelot's theory can of course read this paper with the assumption that the case of interest is the one under discussion.

Recall that the Dirichlet problem has two parts. First, given a connected open set U with $\bar{U}$ compact, associate with each continuous real-valued function f on $\partial U = \bar{U} - U$ a harmonic function $H(f)$ on U so that the mapping $f \to H(f)$ is positive and linear, and so that if we have a superharmonic function v on U (see Section 2) with $\liminf v \geqq f$ on ∂U, then $v \geqq H(f)$ on U. If this can be done and g is continuous on $\bar{U}$ and harmonic on U, then $H(g \mid \partial U) = g \mid U$. Moreover, by the Riesz representation theorem, there is on ∂U a Borel measure, denoted by μ_x^U or just μ_x, for each $x \in U$ such that

$$H(f)(x) = \int_{\partial U} f d\mu_x^U$$

for each continuous f on ∂U. The measure μ_x^U is called harmonic measure for x with respect to U.

The second part of the Dirichlet problem is to determine which points $y \in \partial U$ have the property that

$$\lim_{\substack{x \in U \\ x \to y}} H(f)(x) = f(y)$$

for each continuous f on ∂U. Points with this property are called regular points on ∂U, and if all points on ∂U are regular, then ∂U or just U is called regular. If U is regular, the mapping $f \to H(f)$ is uniquely determined.

In general, integrability with respect to μ_x is independent of the choice of $x \in U$. If $f \geqq 0$ is integrable on ∂U, then its integral, as a function of x, is harmonic and either identically zero or everywhere positive. (See [4] or [22].) Thus the measures μ_x are mutually absolutely continuous; the corresponding Radon–Nikodym derivatives will be central to the results that follow.

As is well known, the open unit disc $D = \{z \in C; |z| < 1\}$ with boundary $\partial D = \{z \in C: |z| = 1\}$ is regular for the Dirichlet problem. For any continuous f on ∂D and any point $a \in D$,

$$H(f)(a) = \frac{1}{2\pi}\int_{|z|=1} \frac{1-|a|^2}{|z-a|^2} f(z)\, d\theta(z);$$

here $d\theta$ refers to Lebesgue measure on ∂D. Martin's generalization of the Poisson Kernel $P(z, a) = (1 - |a|^2)/|z - a|^2$ is obtained as a limit of normalized Green's functions. We shall instead consider a function $q(z, a)$ which is the limit as $b \to z$ of functions which for the disc D and $a, b \in D$ have the form

$$q(b, a) = \frac{1}{2\pi}\int_{|z|=1} P(z, a) P(z, b)\, d\theta(z).$$

We briefly review some important aspects of potential theory for D that will be generalized with the kernel q.

1) If x_0 is the origin and $x \in D$, then $d\mu_{x_0} = (1/2\pi)d\theta$ and $P(\cdot, x) = d\mu_x / d\mu_{x_0}$.

2) If $z \in \partial D$, then $P(z, \cdot)$ is a minimal harmonic function with $P(z, x_0) = 1$. This means that if h is harmonic on D and $0 \leqq h(x) \leqq P(z, x)$ for each $x \in D$, then $h = \lambda P(z, \cdot)$ for some $\lambda \leqq 1$.

3) (Herglotz, 1911) For each positive harmonic function h on D, there is a measure ρ_h on ∂D such that for each $x \in D$,

$$h(x) = \int_{\partial D} P(z, x)\, d\rho_h(z).$$

4) (Fatou, 1906) Each positive harmonic function h on D has finite radial limits $f(z)$ at almost all points $z \in \partial D$ (with respect to Lebesgue measure). If h is bounded or the limit of an increasing sequence of bounded harmonic functions, i.e., quasibounded, then for any Borel set $A \subset \partial D$,

$$\rho_h(A) = \frac{1}{2\pi}\int_A f\, d\theta = \int_A f\, d\mu_{x_0} = \int_A f\, d\rho_1.$$

This paper has six sections. In Section 2 we discuss Brelot's potential theory for a harmonic space W and give nonstandard interpretations of some of the results in that theory. In Section 3 we use a nonstandard internal region $\Omega \subset {}^*W$ with $\partial\Omega$ contained in the monad of the one point compactification of W to obtain a solution of the first part of the Dirichlet problem for any compactification of W. By standardizing a nonstandard measure space on $\partial\Omega$ as in [25], we obtain in Section 4 a new construction of the Martin–Choquet integral representation for positive harmonic functions on W. In Section 5 we review properties of the harmonic part Γ of the Wiener compactification that are needed in Section 6. We also establish a criterion for characterizing points of Γ due to A. Cornea and the author. In Section 6 we obtain a compactification of W with the property that every point of $\Delta = \bar{W} - W$ corresponds to a non-negative harmonic function and almost all points (with respect to harmonic measure) on Δ correspond to positive minimal harmonic functions and are regular. If h is a bounded or quasibounded positive harmonic function on W, then the maximal representing measure for h is supported by the points of Δ corresponding to minimal harmonic functions. A generalization of Fatou's theorem is valid for Δ.

We assume that the reader is familiar with nonstandard analysis (see [33]). We shall be working with a denumerably comprehensive enlargement. This means that if S is a standard set and A_n is internal with $A_n \in {}^*S$ for each $n \in N$, then the external sequence $\{A_n : n \in N\}$ is the restriction to N of an internal function from ${}^*N$ into ${}^*S$. Enlargements which are ultrapowers or $\aleph_1$ saturated models have this property. (See [28, pp. 27–35].)

The notation used here is the same as in [22] and [33] with a few exceptions. The symbol μ is used to denote harmonic measure and $m(a)$ to denote the monad of a. If a and b are in the extension ${}^*R$ of the real numbers R, then $a \simeq b$ means that $a - b \in m(0)$. We write ${}^\circ a$ to denote the unique real number r with $a \simeq r$ if a is finite, i.e., $|a| < n$ for some natural number n. Otherwise, ${}^\circ a = +\infty$ if a is positive and infinite and ${}^\circ a = -\infty$ if a is negative and infinite. As usual, ${}^*N$, ${}^*R$ and ${}^*C$ denote nonstandard extensions of the natural numbers N, the real numbers R and the complex numbers C respectively; R^+ denotes the positive real numbers.

A constant function with value c is denoted by $\mathbf{c}$. If f is a function and A is contained in its domain, then $f \mid A$ denotes the restriction of f to A. Instead of $\lim_{x \in A, x \to x_0} f(x)$, $\liminf_{x \in A, x \to x_0} f(x)$, and $\limsup_{x \in A, x \to x_0} f(x)$, we simply write $\lim_A f(x_0)$, $\liminf_A f(x_0)$, and $\limsup_A f(x_0)$ respectively. Recall for example, that

$$\limsup_A f(x_0) = \inf_{U \in \mathcal{N}(x_0)} \Big(\sup_{x \in U \cap A} f(x) \Big),$$

where $\mathcal{N}(x_0)$ is the family of neighborhoods of x_0. If f and g are real-valued functions with domain B, then $f \wedge g$ and $f \vee g$ are defined for each $x \in B$ by $f \wedge g(x) = \min(f(x), g(x))$ and $f \vee g(x) = \max(f(x), g(x))$. The family of all continuous real-valued functions on a set B is denoted by $C(X)$.

By a region, we mean a connected open set; an *inner* region in a set W is one for which the closure in W is compact. If x is a point in a compact Hausdorff space with topology $\mathcal{T}$, then the monad of x, $m(x)$ is given by $m(x) =$

$\bigcap_{x\in U\in\mathcal{T}} {}^*U$; we write $x = {}^\circ y$ for each $y \in m(x)$. The topology of uniform convergence on compact sets is called the u.c.c. topology. The symbol ▮ denotes the end of a proof.

2. Brelot's potential theory

We shall here review and give some nonstandard interpretations of the potential theory of Marcel Brelot. (See [5] or [6] or [22].) The domain in question is a locally compact Hausdorff space W which is connected and locally connected but not compact. Let $\mathcal{H}$ be a family of real-valued continuous functions called harmonic functions. We assume that each $f \in \mathcal{H}$ has an open domain $\mathcal{D}(f)$ in W and for each open set $\Omega \subset W$, $\mathcal{H}_\Omega = \{f \in \mathcal{H} : \mathcal{D}(f) = \Omega\}$ is a real vector space. We further assume that $\mathcal{H}$ satisfies the following three axioms of Brelot derived in part from previous axiom systems of J. L. Doob and G. L. Tautz (see the second part of [5]) and later generalized by H. Bauer [3] and Constantinescu and Cornea [10].

AXIOM I. *A function g with an open domain $\Omega \subset W$ is an element of $\mathcal{H}$ if for every point $x \in \Omega$ there is an $h \in \mathcal{H}$ and an open set ω such that $x \in \omega \subset \Omega$ and $g \mid \omega = h \mid \omega$.*

AXIOM II. *There is a base for the topology of W consisting of inner regions which are regular for the Dirichlet problem.* (See Section 1.)

AXIOM III. *If $\Omega \subset W$ is a region and $\mathcal{F} \subset \mathcal{H}_\Omega$ is a family directed by increasing order (i.e., $\forall f_1, f_2 \in \mathcal{F}$, $\exists f_3 \in \mathcal{F}$ with $f_3 \geqq f_1 \vee f_2$) then the upper envelope of $\mathcal{F}$ is in $\mathcal{H}$ if it is finite at any point of Ω.*

It follows from Axiom I that the restriction of a harmonic function to an open subset of its domain is harmonic. It is sufficient to assume Axiom III only for increasing sequences of harmonic functions; the general case then follows (see [8]).

As in Section 1, we let μ_x^Ω or just μ_x denote harmonic measure for x and Ω. It follows from Axioms I and II that the uniform limit of harmonic functions is harmonic, since the integral with respect to harmonic measure of such a limit h is equal to h.

Axiom III is called Harnack's principle. In 1964, Gabriel Mokobodski used Axioms I and II and the existence of Radon–Nikodym derivatives of harmonic measures with respect to a fixed harmonic measure to show the equivalence of Harnack's principle (Axiom III) and Harnack's inequality, given below as

Axiom III′. His result was established for those harmonic spaces $(\mathcal{H}, W)$ for which W has a countable base for its topology. Bertram Walsh and the author [26] extended his result to arbitrary Brelot harmonic spaces.

Axiom III′. *If Ω is a region in W, then every nonnegative function in $\mathcal{H}_\Omega$ is either identically equal to 0 or has no zeros in Ω. Furthermore, for any point $x_0 \in \Omega$, the set*

$$\Phi_{x_0}^{\Omega} = \{h \in \mathcal{H}_\Omega : h \geqq \mathbf{0} \quad \text{and} \quad h(x_0) = 1\}$$

is equicontinuous at x_0.

A consequence of Axiom III′ is the fact that for any region Ω and any compact subset $K \subset \Omega$, there is a constant $M \geqq 1$ such that for each $h \geqq 0$ in $\mathcal{H}_\Omega$ and each pair of points x_1 and x_2 in K, the relation

$$\frac{1}{M} \cdot h(x_1) \leqq h(x_2) \leqq M \cdot h(x_1) \tag{1}$$

holds. Moreover, for any point $x \in \Omega$ and any constant $M > 1$, there is a compact neighborhood K of x in which (1) holds.

Associated with a harmonic class $\mathcal{H}$ are the families of superharmonic and subharmonic functions. Recall that a function f is lower semicontinuous on a set A if for each $x \in A$,

$$-\infty < f(x) \leqq \liminf_{A-\{x\}} f(x).$$

Definition. A lower semicontinuous function v with open domain $\Omega \subset W$ is called superharmonic (with respect to $\mathcal{H}$) and we write $v \in \bar{\mathcal{H}}_\Omega$ or just $v \in \bar{\mathcal{H}}$ if $v(x) < +\infty$ for some point x in each component Ω and

$$v(x) \geqq \int_{\partial U} v \, d\mu_x^U$$

for each regular inner region U with $\bar{U} \subset \Omega$ and each $x \in U$. If $-w \in \bar{\mathcal{H}}$, then w is called subharmonic and we write $w \in \underline{\mathcal{H}}$. A superharmonic function whose greatest harmonic minorant is $\mathbf{0}$ is called a potential.

For a local definition of $\bar{\mathcal{H}}$, see [4] or [22, p. 174]. A function h is harmonic if and only if it is superharmonic and subharmonic. If v_1 and v_2 are in $\bar{\mathcal{H}}_\Omega$ for some open $\Omega \subset W$ and $\alpha \in R^+$, $v_1 + v_2$, αv_1 and $v_1 \wedge v_2$ are in $\bar{\mathcal{H}}_\Omega$. To obtain a minimum principle for $\bar{\mathcal{H}}$, and thus a maximum principle for $\underline{\mathcal{H}}$, we shall assume hereafter that $\mathcal{H}$ satisfies the following axiom.

AXIOM IV. *The function* $\mathbf{1}$ *is in* $\bar{\mathcal{H}}_W$.

With this axiom and Axiom III, it follows that if Ω is a region and $v \in \bar{\mathcal{H}}_\Omega$ takes a minimum value α in Ω, then either $v = \boldsymbol{\alpha}$ in Ω or $\alpha > 0$. (See section 2 of [22].)

From this point on, we assume that W and $\mathcal{H}$ are fixed, and we choose a denumerably comprehensive enlargement of a mathematical structure containing the real numbers and W.

Given $x_0 \in W$, it is well known that the family

$$\Phi'_{x_0} = \{h \in \mathcal{H}_W : h \geqq 0, \quad h(x_0) \leqq 1\}$$

is compact in the u.c.c. topology. We shall need a nonstandard interpretation of a slightly more general result. Recall Robinson's fundamental theorem ([33, p. 93]), that a topological space A is compact if and only if for each $y \in {}^*A$ there is an $x \in A$ with $y \in m(x)$.

PROPOSITION 2.1. *Given a region* $\Omega \subset W$, *a point* $x_0 \in \Omega$, *numbers* $m \geqq 0$ *and* $M \geqq 0$ *in* R, *let*

$$\mathcal{F} = \{h \in \mathcal{H}_\Omega : -\boldsymbol{m} \leqq h \quad \text{on} \quad \Omega \quad \text{and} \quad h(x_0) \leqq M\}.$$

Then $\mathcal{F}$ *is equicontinuous on* Ω *and compact with respect to the u.c.c. topology which is the same as the topology of pointwise convergence on* $\mathcal{F}$. *That is, given* f *in the non-standard extension* ${}^*\mathcal{F}$ *of* $\mathcal{F}$ *there is a standard* $h \in \mathcal{F}$ *such that on each standard compact* $K \subset \Omega$

$$\sup_{y \in {}^*K} |f(y) - {}^*h(y)| \simeq 0.$$

PROOF. Fix $x_1 \in \Omega$. By corollary 4.2 of [22], there is a regular inner region U with $x_0 \in U$, $x_1 \in U$ and $\bar{U} \subset \Omega$. Let $H(\mathbf{1})$ be the function which is continuous on $\bar{U}$ and harmonic on U with $H(\mathbf{1}) \mid \bar{U} - U = \mathbf{1}$. Then $c = \min_{x \in \bar{U}} H(\mathbf{1}) > 0$. By Harnack's inequality, the family

$$\left\{h \mid U + \frac{m}{c} H(\mathbf{1}) : h \in \mathcal{F}\right\}$$

is bounded and equicontinuous at x_1. Therefore the family $\mathcal{F}$ itself is bounded and equicontinuous at the arbitrary point $x_1 \in \Omega$ and thus at every $x \in \Omega$.

Let $\mathcal{T}_p$ and $\mathcal{T}_{\text{u.c.c.}}$ denote the topology of pointwise convergence and the u.c.c. topology respectively on the space of real-valued functions on Ω. The $\mathcal{T}_p$-closure of $\mathcal{F}$, $\bar{\mathcal{F}}^p$, is equicontinuous on Ω. It is well known and easy to prove that $\mathcal{T}_p$ and

$\mathcal{T}_{\text{u.c.c.}}$ are the same on an equicontinuous family such as $\bar{\mathcal{F}}^p$. Since $\bar{\mathcal{F}}^p$ is $\mathcal{T}_p$-compact, it is $\mathcal{T}_{\text{u.c.c.}}$-compact. Any point in $\bar{\mathcal{F}}^p$ is the uniform limit of harmonic functions and is, therefore, in $\mathcal{F}$. Thus $\mathcal{F} = \bar{\mathcal{F}}^p$ is compact. ∎

Let Ω be a region in W, and let f be an element of ${}^*\mathcal{H}_\Omega$ such that f is bounded below by some standard real number and f is finite at some standard point $x_0 \in \Omega$. Let h be defined on Ω by setting $h(x) = {}^\circ(f(x))$ for each $x \in \Omega$. We have just shown that $h \in \mathcal{H}_\Omega$; we shall call h the standard part of f and write $h = {}^\circ f$.

A countable exhaustion of W is an increasing sequence of sets $A_1 \subset A_2 \cdots \subset A_n \cdots$, with $W = \bigcup_{n=1}^\infty A_n$. Using results in [8] and the validity of Harnack's inequality in an arbitrary harmonic space [26], A. Cornea has established the existence of a countable exhaustion of W by compact sets. It follows that if $x_0 \in W$ and

$$\Phi'_{x_0} = \{h \in \mathcal{H}_W : h \geqq 0 \quad \text{and} \quad h(x_0) \leqq 1\}$$

then Φ'_{x_0} with the u.c.c. topology is a metric space. Here we may let the metric $d = \sum_{n=1}^\infty (1/(2^n c_n)) d_n$, where for each $n \in N$, K_n is the nth compact set in a countable exhaustion of W by compact sets,

$$c_n = 2 \sup_{y \in K_n, h \in \Phi'_{x_0}} |h(y)|,$$

and for $f, h \in \Phi'_{x_0}$,

$$d_n(f, h) = \sup_{y \in K_n} |f(y) - h(y)|.$$

It also follows from Cornea's result and a theorem due to R.-M. Hervé and the author ([22, p. 184]), that there is a countable exhaustion of W by regular inner regions in W. Given such an exhaustion $\{\Omega_n : n \in N\}$, to what use can we put the existence of Ω_γ where $\gamma \in {}^*N - N$? For this paper we will only use the fact that the first part of the Dirichlet problem is solvable for an inner region.

Definition. Let Ω be a standard region in W with closure $\bar{\Omega}$ in some fixed compactification of W. Let f be a bounded real-valued function on $\bar{\Omega} - \Omega$. The upper Brelot–Wiener–Perron envelope $\bar{H}(f, \Omega)$ of f is the lower envelope of the set

$$\{v \in \bar{\mathcal{H}}_\Omega : \liminf_{\Omega} v(x) \geqq f(x) \ \forall x \in \bar{\Omega} - \Omega\}.$$

The lower envelope $\underline{H}(f, \Omega)$ is $-\bar{H}(-f, \Omega)$. Since $1 \in \bar{\mathcal{H}}_W$, $\underline{H}(f, \Omega) \leqq \bar{H}(f, \Omega)$. We say that f is resolutive if $\bar{H}(f, \Omega) = \underline{H}(f, \Omega)$, and we say that $\bar{\Omega} - \Omega$ is resolutive if each $f \in C(\bar{\Omega} - \Omega)$ is, in which case $H(f, \Omega)$ denotes the unique solution

$\bar{H}(f,\Omega)=\underline{H}(f,\Omega)$ of the first part of the Dirichlet problem for Ω as defined in Section 1.

PROPOSITION 2.2 (Brelot, Hervé). *If Ω is an inner region in W, then $\bar{\Omega}-\Omega$ is resolutive.*

PROOF. See Hervé [18, lemma 6.1] and Brelot [4, p. 111]. That the hypothesis of Hervé's result is satisfied, i.e., that there is a positive potential defined on $\bar{\Omega}$, has been established by the author in [22, theor. 6.8]. ∎

3. A solution of the Dirichlet problem

In this section we generalize N. Wiener's solution of the Dirichlet problem [36], [37] to arbitrary compactifications (e.g., the Stone–Čech compactification) of the harmonic space W. For resolutive compactifications of W, the solution agrees with that given by the Brelot–Wiener–Perron method defined in Section 2.

DEFINITION. If Ω is an internal inner region in $^*W$ and $^*K\subset\Omega$ for each standard compact set $K\subset W$, then we shall say that the boundary of Ω is contained in the monad of ∞ and write $\partial\Omega\subset m(\infty)$.

By the results of Section 2, there exists an internal, regular inner region Ω in $^*W$ with $\partial\Omega\subset m(\infty)$. For example, if $W=\{z\in C:|z|<1\}$, let $\Omega=\{z\in {}^*C:|z|<1-\delta\}$ where $\delta\simeq 0$, and $\delta>0$.

THEOREM 3.1. *Let Ω be an internal inner region in $^*W$ with $\partial\Omega\subset m(\infty)$. Let $\bar{W}$ be an arbitrary compactification of W, and for each $f\in C(\bar{W}-W)$ let $\hat{f}$ be a continuous extension of f to $\bar{W}$* (see [34, p. 148]). *For each $x\in W$, let*

$$h_f(x)={}^\circ(H(^*\hat{f}\mid\partial\Omega,\Omega)(x));$$

i.e., h_f is the standard part of the internal solution of the Dirichlet problem for $^*\hat{f}$ on $\partial\Omega$. Then the mapping $f\to h_f$ is a well defined, positive linear operator from $C(\bar{W}-W)$ into $\mathscr{H}_W$ with $\underline{H}(f,W)\leqq h_f\leqq\bar{H}(f,W)$ for each $f\in C(\bar{W}-W)$.

PROOF. Given f and $\hat{f}$, let $\tilde{f}$ be another continuous extension of f to $\bar{W}$. For each $y\in\partial\Omega$, there is a unique $z\in\bar{W}-W$ with $y\in m(z)$, and so $^*\tilde{f}(y)\simeq f(z)\simeq {}^*\hat{f}(y)$. Therefore $\sup_{y\in\partial\Omega}|\tilde{f}(y)-\hat{f}(y)|=\delta\simeq 0$, and since $\mathbf{1}\in\bar{\mathscr{H}}_W$, we have for each $x\in\Omega$

$$\left|\int_{\partial\Omega}{}^*\tilde{f}\,d\mu_x^\Omega-\int_{\partial\Omega}{}^*\hat{f}\,d\mu_x^\Omega\right|\leqq\delta\int_{\partial\Omega}d\mu_x^\Omega\leqq\delta\simeq 0.$$

It follows that h_f is well defined. By Axiom I and Proposition 2.1, h_f is harmonic on W, and it is easy to see that the mapping $f \to h_f$ is positive and linear.

Let v be a superharmonic function on W with $\liminf_W v(z) \geqq f(z)$ for each $z \in \bar{W} - W$. Given $\varepsilon > 0$, each point $z \in \bar{W} - W$ is contained in an open neighborhood U_z such that $v + \varepsilon - \hat{f} > 0$ on $U_z \cap W$. Since $v + \varepsilon - \hat{f}$ is lower semicontinuous on W, the set $K = \{x \in W : v + \varepsilon - \hat{f} \leqq 0\}$ is compact, and therefore $K \subset \Omega$. Thus $*v + \varepsilon > *\hat{f}$ on $\partial\Omega$, and so at each $x \in W$, $v(x) + \varepsilon \geqq h_f(x)$. It follows, since ε is arbitrary, that for each $f \in C(\bar{W} - W)$, $\bar{H}(f, W) \geqq h_f$, whence $\underline{H}(f, W) \leqq h_f$ on W. ∎

If $\bar{W}$ is a resolutive compactification of W, then $h_f = \bar{H}(f, W) = \underline{H}(f, W)$; i.e., h_f is the unique solution of the Dirichlet problem for each $f \in C(\bar{W} - W)$. The Wiener compactification discussed in Section 5 is the largest resolutive compactification of W.

Given Theorem 3.1, we see that the monad of a regular point is "unbroken" in the sense made precise by the following result.

THEOREM 3.2. *Given a resolutive compactification $\bar{W}$ of W let z be a regular point on $\bar{W} - W$. If $U \subset \bar{W}$ is an open neighborhood of z and Ω is an internal inner region in $*W$ with $\partial\Omega \subset m(\infty)$, then $*U \cap \partial\Omega \neq \emptyset$. It follows that if z has a countable base for its neighborhood system or if our enlargement is $\aleph$-saturated* (see [28]) *where the cardinal number $\aleph$ is greater than the cardinality of the neighborhood system of z, then the monad of z contains a point of $\partial\Omega$ for each internal inner region $\Omega \subset *W$ with $\partial\Omega \subset m(\infty)$. In any case, if $\{\Omega_n\}$ is a countable exhaustion of W by standard inner regions and U is a neighborhood of z, there is an $n_0 \in N$ such that for all $n \geqq n_0$, $\partial\Omega_n \cap U \neq \emptyset$.*

PROOF. Given U, we may choose f to be a continuous function on $\bar{W}$ so that $f(z) = 1$ and $f(x) = 0$ for all $x \notin U$. Since z is regular, $\lim_W h_f(z) = 1$. If there is an internal inner region Ω with $\partial\Omega \subset m(\infty)$ and $\partial\Omega \cap *U = \emptyset$, then

$$h_f = {}^\circ H(*f \mid \partial\Omega, \Omega) = {}^\circ H(\mathbf{0}, \Omega) = \mathbf{0},$$

but this is impossible. It follows that for a standard exhaustion $\{\Omega_n\}$ of W by inner regions, if there is an $n \in *N$ such that $*U \cap \partial\Omega_n = \emptyset$, then there is a last one, and it is standard. The rest follows from the definition of saturation [28] and the assumption that our enlargement is at least denumerably comprehensive, i.e., $\aleph_1$-saturated. ∎

In Section 5 we establish a converse of Theorem 3.2 due to A. Cornea and the author for the Wiener compactification of W.

4. A construction of maximal representing measures for positive harmonic functions

In this section, x_0 is a fixed point in W, $\mathcal{H}_W^+ = \{h \in \mathcal{H}_W : h \geqq 0\}$ and $\Phi_{x_0} = \{h \in \mathcal{H}_W^+ : h(x_0) = 1\}$. It was shown in Section 2 that Φ_{x_0} as a subspace of $\mathcal{H}_W^+ - \mathcal{H}_W^+$ equipped with the u.c.c. topology is a metric space. Clearly Φ_{x_0} is a compact, convex subset of $\mathcal{H}_W^+ - \mathcal{H}_W^+$. The set $\mathcal{E}_{x_0}$ of extreme elements of Φ_{x_0} is G_δ in Φ_{x_0}. (See [32, prop. 1.3].) A function $h \in \Phi_{x_0}$ is in $\mathcal{E}_{x_0}$ if and only if it is a minimal harmonic function; i.e., given $g \in \mathcal{H}_W^+$, if $g \leqq h$ then $g = \lambda h$ for some $\lambda \geqq 0$. Moreover Φ_{x_0} is a Choquet simplex; i.e., there is a greatest harmonic minorant of $f \wedge g$ for any pair $f, g \in \mathcal{H}_W^+$. (See chapter 9 of [32].)

Let $\mathcal{P}$ denote the set of probability measures defined on the Borel subsets of Φ_{x_0}. Given a measure $\lambda \in \mathcal{P}$, there is a unique element $h \in \Phi_{x_0}$ such that for each $x \in W$,

$$h(x) = \int_{\Phi_{x_0}} T_x(g)\, d\lambda(g)$$

where $T_x(g) = g(x)$ for each $g \in \Phi_{x_0}$. (See [32].) We say that λ represents h, and for any continuous linear functional F on $\mathcal{H}_W^+ - \mathcal{H}_W^+$ we have $\int_{\Phi_{x_0}} F(g)\, d\lambda(g) = F(h)$. If $h \in \mathcal{E}_{x_0}$, only δ_h represents h, where $\delta_h(\Phi_{x_0}) = 1$ and $\delta_h(\Phi_{x_0} - \{h\}) = 0$. (See proposition 1.4 of [32].)

A real-valued function ψ is convex on Φ_{x_0} if $\psi(\alpha f + (1-\alpha)g) \leqq \alpha\psi(f) + (1-\alpha)\psi(g)$ for each pair f, g in Φ_{x_0} and each $\alpha \in R$ with $0 \leqq \alpha \leqq 1$. Given λ and ν in $\mathcal{P}$, we write $\lambda < \nu$ if for each continuous convex function ψ on Φ_{x_0} we have

$$\int_{\Phi_{x_0}} \psi(g) d\lambda(g) \leqq \int_{\Phi_{x_0}} \psi(g) d\nu(g).$$

When λ and ν both represent the same $h \in \Phi_{x_0}$, we write $\lambda \sim \nu$. If $\lambda < \nu$ then $\lambda \sim \nu$ since evaluation, T_x, at a point x and $-T_x$ are convex on Φ_{x_0}. If $\lambda < \nu$ and $\nu < \lambda$, then $\nu = \lambda$ since the family of differences of continuous convex functions is uniformly dense in $C(\Phi_{x_0})$.

PROPOSITION 4.1 (Choquet). *For each $h \in \Phi_{x_0}$, there is a unique probability measure ρ_h on Φ_{x_0} such that $\rho_h(\Phi_{x_0} - \mathcal{E}_{x_0}) = 0$ and ρ_h represents h. If $\nu \in \mathcal{P}$ also represents h then $\nu < \rho_h$; that is, ρ_h is the unique maximal representing measure for h with respect to the ordering $<$.*

PROOF. See [32, chaps. 3, 4, and 9]. ∎

Fundamental for this paper is a special case of a corollary due to Beno Fuchssteiner [14] of a result of Cartier, Fell and Meyer (see [1, p. 23]). We state as Proposition 4.2 the version of the latter result needed here. The proof given of Proposition 4.2 and the first corollary that follows were communicated to the author by B. Fuchssteiner.

DEFINITION. An affine decomposition in S of a vector x in a subset S of a vector space X is a sum $x = \Sigma_{i=1}^n \alpha_i x_i$ with $\Sigma_{i=1}^n \alpha_i = 1$, $x_i \in S$, and $0 \leqq \alpha_i < 1$ for each $i = 1, 2, \cdots, n$. A mapping M is called affine on S if for each affine decomposition $x = \Sigma_{i=1}^n \alpha_i x_i$ in S of an element $x \in S$ we have $M(x) = \Sigma_{i=1}^n \alpha_i M(x_i)$.

PROPOSITION 4.2 (Cartier, Fell, Meyer). *Given λ and ν in $\mathcal{P}$, suppose that for every affine decomposition $\lambda = \Sigma_{i=1}^n \alpha_i \lambda_i$ of λ in $\mathcal{P}$ there is an affine decomposition $\nu = \Sigma_{i=1}^n \alpha_i \nu_i$ of ν in $\mathcal{P}$ with $\nu_i \sim \lambda_i$ for each $i = 1, \cdots, n$. Then $\lambda < \nu$.*

PROOF. Let $F_1, \cdots, F_n$ be an arbitrary finite set of functions of the form $F + r$ where F is a continuous linear functional on $\mathcal{H}_W^+ - \mathcal{H}_W^+$ and $r \in R$. Let G be the convex function on Φ_{x_0} defined by setting $G(h) = \max_{1 \leqq i \leqq n} F_i(h)$ for each $h \in \Phi_{x_0}$. We need only show that $\int_{\Phi_{x_0}} G(h)\, d\lambda(h) \leqq \int_{\Phi_{x_0}} G(h)\, d\nu(h)$, since the set of all such G's is uniformly dense in the set of continuous convex functions on Φ_{x_0}. (See [1, pp. 1–3].) For each i, $1 \leqq i \leqq n$, let $X_i = \{h \in \Phi_{x_0} : G(h) = F_i(h)\}$, $Y_i = X_i - \bigcup_{j<i} X_j$, and $\alpha_i = \lambda(Y_i)$. If $\alpha_i = 0$, let $\lambda_i = \lambda$; otherwise, let $\alpha_i \lambda_i$ be the restriction of λ to Y_i. Each λ_i represents some $h_i \in \Phi_{x_0}$. Since $\lambda = \Sigma_{i=1}^n \alpha_i \lambda_i$ there is for each i, $1 \leqq i \leqq n$, a $\nu_i \in \mathcal{P}$ such that $\nu_i \sim \lambda_i$ and $\Sigma_{i=1}^n \alpha_i \nu_i = \nu$. Therefore

$$\begin{aligned}\int_{\Phi_{x_0}} G\, d\lambda &= \sum_{i=1}^n \alpha_i \int_Y G\, d\lambda_i = \sum_{i=1}^n \alpha_i \int_{Y_i} F_i\, d\lambda_i \\ &= \sum_{i=1}^n \alpha_i \int_{\Phi_{x_0}} F_i\, d\lambda_i = \sum_{i=1}^n \alpha_i F_i(h_i) = \sum_{i=1}^n \alpha_i \int_{\Phi_{x_0}} F_i\, d\nu_i \\ &\leqq \sum_{i=1}^n \alpha_i \int_{\Phi_{x_0}} G\, d\nu_i = \int_{\Phi_{x_0}} G\, d\nu. \end{aligned}$$ ∎

COROLLARY 4.3 (B. Fuchssteiner [14]). *If M is an affine mapping of Φ_{x_0} into $\mathcal{P}$ such that for each $h \in \Phi_{x_0}$, $M(h)$ represents h, then for each $h \in \Phi_{x_0}$, $M(h)$ is the maximal representing measure for h.*

PROOF. Given $h \in \Phi_{x_0}$, assume that $M(h) < \lambda$ for some $\lambda \in \mathcal{P}$; we must show that $\lambda = M(h)$. Let $\lambda = \Sigma_{i=1}^n \alpha_i \lambda_i$ be any affine decomposition of λ in $\mathcal{P}$. Then λ_i represents some $h_i \in \Phi_{x_0}$ for each i, $1 \leqq i \leqq n$. For each $x \in W$,

$$h(x)=\int_{\Phi_{x_0}} T_x\,d\lambda=\sum_{i=1}^{n}\alpha_i\int_{\Phi_{x_0}} T_x\,d\lambda_i=\Sigma_{i=1}^{n}\alpha_i h_i(x).$$

That is, $h=\Sigma_{i=1}^{n}\alpha_i h_i$, so $M(h)=\Sigma_{i=1}^{n}\alpha_i M(h_i)$. Thus $\lambda < M(h)$, whence $\lambda = M(h)$. ∎

COROLLARY 4.4. *If M' is an affine mapping from the set $\Phi^b_{x_0}$ of bounded elements of Φ_{x_0} into $\mathcal{P}$ such that $M'(h)$ represents h for each $h\in\Phi^b_{x_0}$, then $M'(h)$ is the maximal representing measure for each $h\in\Phi^b_{x_0}$.*

PROOF. The proof is the same as the one for Corollary 4.3 except we note that if $h_i\in\Phi_{x_0}$ and $0<\alpha_i\leqq 1$ for each i, $1\leqq i\leqq n$, then $h=\Sigma_{i=1}^{n}\alpha_i h_i$ is bounded if and only if each h_i is bounded, since $(1/\alpha_i)h\geqq h_i$ for $1\leqq i\leqq n$. ∎

We now choose and fix an internal inner region $\Omega\subset W^*$ with $\partial\Omega\subset m(\infty)$, and we let $\mu_x=\mu_x^{\Omega}$ for each $x\in\Omega$. If $h\in\Phi_{x_0}$ and $x\in W$, then $h(x)=\int_{\partial\Omega}{}^*h\,d\mu_x$. We shall use this fact to construct the maximal representing measure for h; it is a standard form of $h\,d\mu_{x_0}$.

Recall that a *finite set is an internal set in one-one correspondence with an initial segment of $^*N$. Such a set has the formal properties of a finite set. There exists a *finite collection $\{A_i: 1\leqq i\leqq\gamma\}$ of disjoint, internal Borel measurable sets in $\partial\Omega$ with $\bigcup_{i=1}^{\gamma}A_i=\partial\Omega$ such that for each standard $f\in C(W)$, $\sup_{A_i}{}^*f-\inf_{A_i}{}^*f\simeq 0$ for each i, $1\leqq i\leqq\gamma$. To show this, we imbed $C(W)$ in a *finite collection $\mathcal{G}$ of $^*C(W)$. Given $f\in\mathcal{G}$ and $\delta>0$ with $\delta\simeq 0$, let P_f be the inverse image under f of a *finite partition of the range of $f\mid\partial\Omega$ into intervals of length smaller than δ. The common refinement of $\{P_f: f\in\mathcal{G}\}$ is the desired partition of $\partial\Omega$.

We now let $\tilde{X}=\{A_i\subset\partial\Omega: 1\leqq i\leqq\omega\}$ be a fixed *finite collection of disjoint, internal Borel measurable subsets of $\partial\Omega$ such that $\mu_{x_0}(A_i)>0$ and $\sup_{A_i}{}^*f-\inf_{A_i}{}^*f\simeq 0$ for each $f\in C(W)$ and each i, $1\leqq i\leqq\omega$, and such that $\mu_{x_0}(\partial\Omega-\bigcup_{i=1}^{\omega}A_i)=0$. For each $A_i\in\tilde{X}$, $\mu_x(A_i)$ as a function of x is the solution of the internal Dirichlet problem with respect to the characteristic function of A_i on $\partial\Omega$; let g_i be defined by setting $g_i(x)=\mu_x(A_i)/\mu_{x_0}(A_i)$ for each $x\in\Omega$. Let $X=\{g_i: 1\leqq i\leqq\omega\}$, let S be the mapping from X into Φ_{x_0} such that $S(g_i)={}^\circ g_i$ for each i, $1\leqq i\leqq\omega$, and let $Y\subset\Phi_{x_0}$ be the image $S[X]$. Let $\mathcal{A}$ be the algebra (in the usual sense) of internal sets in X, let $\mathcal{M}_X$ be the smallest (external) σ-algebra of subsets of X such that $\mathcal{M}_X\supset\mathcal{A}$, and let $\mathcal{M}_Y$ be the σ-algebra in Y consisting of sets B such that $S^{-1}[B]\in\mathcal{M}_X$.

PROPOSITION 4.5. *The set $Y \subset \Phi_{x_0}$ is compact in the u.c.c. topology and each Borel set in Y is an element of $\mathcal{M}_Y$.*[†]

PROOF. Let $\{h_n\}$ be a sequence in Y, and let $\{K_n\}$ be an exhaustion of W by compact sets. By taking a subsequence of $\{h_n\}$, we may assume without loss of generality that there is an $h \in \Phi_{x_0}$ such that for each $n \in N$, $\sup_{K_n} |h - h_n| < 1/(2n)$. We must show that $h \in Y$. Choose for each $n \in N$ a function $f_n \in X$ such that ${}^\circ f_n = h_n$. Let $\{f_n : n \in {}^*N\}$ be a internal extension of the sequence $\{f_n : n \in N\}$. An internal property that holds for each $n \in N$ holds for some $\gamma \in {}^*N - N$. Thus there is a $\gamma \in {}^*N - N$ such that $f_\gamma \in X$ and $\sup_{K_\gamma} |{}^*h - f_\gamma| < 1/\gamma$. Therefore, $h = {}^\circ f_\gamma \in Y$. (Alternatively, $Y = S[X]$ is compact by Robinson's theorem 4.3.12 in [33].)

We must next show that a basic open set in Y is $\mathcal{M}_Y$-measurable; it then follows that the Borel sets in Y are $\mathcal{M}_Y$-measurable. (See [21, p. 49].) Given $\varepsilon > 0$, K compact in W, and $h \in Y$, we will show that

$$B = \{g \in Y : \max_{x \in K} |g(x) - h(x)| < \varepsilon\} \in \mathcal{M}_Y.$$

Choose $f \in X$ so that $h = {}^\circ f$. The set

$$E_n = \left\{\tilde{g} \in X : \max_{x \in {}^*K} |\tilde{g}(x) - f(x)| < \varepsilon - \frac{1}{n}\right\}$$

is internal for each $n \in N$, so

$$\bigcup_{n \in N} E_n = \left\{\tilde{g} \in X : \sup_{x \in {}^*K} |{}^\circ(\tilde{g}(x)) - {}^\circ(f(x))| < \varepsilon\right\} \in \mathcal{M}_X,$$

and $S^{-1}[B] = \bigcup_{n \in N} E_n$. ∎

We now choose and fix $y_i \in A_i$ for each $A_i \in \tilde{X}$. Note that for each $x \in W$

$$h(x) = \int_{\partial\Omega} {}^*h(y)\, d\mu_x(y) \simeq \sum_{i=1}^{\omega} {}^*h(y_i)\, \mu_x(A_i)$$

$$\simeq \sum_{i=1}^{\omega} {}^*h(y_i)\, \mu_{x_0}(A_i) \frac{\mu_x(A_i)}{\mu_{x_0}(A_i)}.$$

That is, h is essentially an affine combination of the functions

$$g_i = \frac{\mu_x(A_i)}{\mu_{x_0}(A_i)} \quad \text{in} \quad X.$$

[†] (Added in proof, April 1976.) A recent result due to Ward Henson shows that $B \subset Y$ is analytic iff $B = S[A]$ for some $A \in \mathcal{M}_Y$. Therefore, $\mathcal{M}_Y$ is the family of Borel subsets of Y.

THEOREM 4.6. *Given $h \in \Phi_{x_0}$, let ν_h be the finitely additive real-valued measure on the algebra $\mathcal{A}$ of internal subsets of X such that for each set $E \in \mathcal{A}$,*

$$\nu_h(E) = {}^0\sum_{i: g_i \in E} {}^*h(y_i)\,\mu_{x_0}(A_i).$$

Then ν_h has a unique countably additive extension, which we also denote by ν_h, to $\mathcal{M}_X$. For each $B \in \mathcal{M}_Y$, let $\tilde{\nu}_h(B) = \nu_h(S^{-1}[B])$. Then $\tilde{\nu}_h$ is the maximal representing measure for h on Φ_{x_0}.

PROOF. That ν_h is uniquely determined and countably additive on $\mathcal{M}_X$ follows from theorem 1 of [25]. Given $x \in \Omega$ and letting T_x denote evaluation at x, we have

$$\begin{aligned} {}^*h(x) &= \int_{\partial\Omega} {}^*h\, d\mu_x \simeq \sum_{i=1}^{\omega} {}^*h(y_i)\,\mu_{x_0}(A_i)\frac{\mu_x(A_i)}{\mu_{x_0}(A_i)} \\ &= \sum_{i=1}^{\omega} T_x(g_i)\,{}^*h(y_i)\,\mu_{x_0}(A_i) \simeq \int_X {}^\circ(T_x(g))\, d\nu_h(g) \end{aligned}$$

by corollary 1 of [25]. Of course, ${}^\circ(T_x(g)) = {}^\circ g(x) = S(g)(x)$ if $x \in W$. Since only real numbers are involved, we have for each $x \in W$,

$$h(x) = \int_X {}^\circ(T_x(g))\, d\nu_h(g) = \int_Y T_x(f)\, d\tilde{\nu}_h(f).$$

Thus $\tilde{\nu}_h$ represents h for each $h \in \Phi_{x_0}$. Since the mapping $h \to \tilde{\nu}_h$ is affine, $\tilde{\nu}_h$ is the maximal representing measure for h for each $h \in \Phi_{x_0}$ by Corollary 4.3. ∎

COROLLARY 4.7. *If $h \in \Phi_{x_0}$ is minimal, then $h \in Y$.*

PROOF. Only unit mass δ_h at h represents h. ∎

As is usual in nonstandard analysis, results such as Theorem 4.6 yield standard limit theorems of the type we now consider. Recall that a statement that can be interpreted in both a standard structure S and a nonstandard extension of S is either true for both or false for both.

THEOREM 4.8. *Fix $h \in \mathscr{E}_{x_0}$, a compact $K_0 \subset W$, and an $\varepsilon > 0$ in R. "There is a compact set $K \supset K_0$ such that on the boundary of each inner region $\Omega \supset K$ there is a Borel measurable set A_Ω with $\mu_{x_0}^\Omega(A_\Omega) > 0$ and*

$$\sup_{x \in K_0} \left| h(x) - \frac{\mu_x(A_\Omega)}{\mu_{x_0}(A_\Omega)} \right| < \varepsilon."$$

PROOF. By Theorem 4.6, the statement in quotes is true for some nonstandard compact set K in ${}^*W$ with $K \supset {}^*K_s$ for every standard compact set K_s in W. Therefore there is a standard compact K for which the statement in quotes is true. ∎

THEOREM 4.9. *Let $h_1, h_2, \cdots, h_n$ be a finite subset of Φ_{x_0}. Fix a compact set $K_0 \subset W$, a Borel set $B \subset \mathscr{E}_{x_0}$, and an $\varepsilon > 0$ in R. "There is a compact set $K \supset K_0$ such that for each inner region $\Omega \supset K$ there is a set $A \subset \partial\Omega$ with*

$$\sup_{x \in K_0} \left| \int_B T_x(f)\, d\rho_{h_j}(f) - \int_A h_j(y)\, d\mu_x^{\Omega}(y) \right| < \varepsilon$$

for $1 \leqq j \leqq n$." Here ρ_{h_j} is the maximal representing measure for h_j on $\mathscr{E}_{x_0}$.

PROOF. Given Ω as in Theorem 4.6, we use the notations established in Theorem 4.6. Let $\hat{B} = S^{-1}[B]$. By theorem 1 of [25], there is an internal union A of elements of $\tilde{X}$ such that if $\hat{A} = \{g_i \in X : A_i \subset A\}$, then $\nu_{h_j}((\hat{B} - \hat{A}) \cup (\hat{A} - \hat{B})) = 0$ for $1 \leqq j \leqq n$. For each $x \in K_0$ and each j, $1 \leqq j \leqq n$, we have by corollary 1 of [25],

$$\int_B T_x(f)\, d\rho_{h_j}(f) = \int_{\hat{B}} {}^\circ(T_x(g))\, d\nu_{h_j}(g)$$

$$= \int_{\hat{A}} {}^\circ(T_x(g))\, d\nu_{h_j}(g) \simeq \sum_{g_i \in \hat{A}} T_x(g_i)^* h_j(y_i)\, \mu_{x_0}(A_i)$$

$$= \sum_{A_i \subset A} \frac{\mu_x(A_i)}{\mu_{x_0}(A_i)} {}^*h_j(y_i)\, \mu_{x_0}(A_i) \simeq \int_A {}^*h_j(y)\, d\mu_x(y).$$

By Harnack's inequality, it follows that for $1 \leqq j \leqq n$,

$$\sup_{x \in {}^*K_0} \left| \int_{{}^*B} {}^*T_x(g)\, d\, {}^*\rho_{h_j}(g) - \int_A {}^*h_j(y)\, d\mu_x(y) \right| < \varepsilon.$$

Since the part of the statement of the theorem in quotes is thus true for some nonstandard compact $K \subset {}^*W$, it is true for some standard compact $K \subset W$. ∎

5. Compactifications for which bounded harmonic functions have continuous extensions

In this section we describe a resolutive compactification $\bar{W}$ of W for which the Radon–Nikodym derivatives of harmonic measures are continuous and every bounded harmonic function and every positive harmonic function has a continuous extension to $\bar{W} - W$. The smallest compactification with these properties is one between the Feller and Wiener compactifications. Here, the ordering is the usual one for compactifications, i.e., $\bar{W} \geqq \tilde{W}$ if there is a continuous map from $\bar{W}$ onto $\tilde{W}$ which is the identity on W. Since we shall give a converse of Theorem 3.2 for the Wiener compactification, and since the analysis is essentially the same for all compactifications between the Feller and Wiener compactifications, all of these compactifications will be considered at the outset. For the original work on this topic in the axiomatic setting see Constantinescu and Cornea [9], Loeb and Walsh [27], and Magea [30]; also see [7]. We follow the development in [27].

Given a collection Q of continuous, bounded, real-valued functions on W, Constantinescu and Cornea [7] have shown that there is a unique (up to homeomorphism) compactification $\bar{W}^Q$ of W such that every function in Q has a continuous extension to $\bar{W}^Q$ and these extensions separate the points of $\bar{W}^Q - W$. One may, following [7], adjoint $C_0(W)$, the continuous functions with compact support in W, to Q and let $\bar{W}^Q$ be the closure of the canonical image of W in the product space

$$\prod_{f \in Q \cup C_0(W)} \left[\inf_{x \in W} f(x),\ \sup_{x \in W} f(x) \right].$$

A method which "works" for arbitrary Hausdorff spaces is given in [23] and [24].

Using nonstandard analysis, one can obtain $\bar{W}^Q$ by letting $m(\infty)$ denote the monad of the one point compactification of W (as before) and calling two points $x, y \in m(\infty)$ equivalent when ${}^*f(x) \simeq {}^*f(y)$ for all $f \in Q$. The points of $\bar{W}^Q - W$ are the equivalence classes $[x]$ and a neighborhood base for $[x]$ consists of sets O determined by a finite set $\{f_1, \cdots, f_n\} \subset Q$, a compact $K \subset W$, and an $\varepsilon > 0$ in R with

$$O = \{y \in W - K : |f_i(y) - {}^\circ({}^*f_i(x))| < \varepsilon,\ 1 \leqq i \leqq n\}$$

$$\cup \{[z] \in \bar{W}^Q - W : {}^\circ|{}^*f_i(z) - {}^*f_i(x)| < \varepsilon,\ 1 \leqq i \leqq n\}.$$

The justification for this method of obtaining $\bar{W}^Q$ is the same as that given by Gonshore for the Stone–Čech compactification [15].

Recall that a potential on W is a superharmonic function on W with greatest harmonic minorant equal to $\mathbf{0}$. We assume throughout the rest of this paper that there is a positive potential and a bounded harmonic function not identically equal to 0, defined on W. This means that $\mathbf{1}$ is not a potential, and $\mathcal{H}$ is hyperbolic on W in the sense of [22]. (See prop. 5.5 and theor. 5.8 of [22].) If there were no potential on W, then the space of bounded functions in $\mathcal{H}_W$ would consist only of multiples of $\mathbf{1}$.

Let P_W denote the set of potentials on W, and let $B\mathcal{H}_W$ denote the bounded harmonic functions on W. $B\mathcal{H}_W$ is a Banach lattice with respect to the sup norm and the lattice operations $\vee_{\mathcal{H}}$ and $\wedge_{\mathcal{H}}$ where $f\wedge_{\mathcal{H}}g$ is the greatest harmonic minorant $\boldsymbol{H}(f\wedge g)$ of $f\wedge g$ and $f\vee_{\mathcal{H}}g = -\boldsymbol{H}(-f\wedge -g)$. The function $\boldsymbol{H}(\mathbf{1})$ is an order unit for $B\mathcal{H}_W$. Here $\boldsymbol{H}(v)$ denotes the greatest harmonic minorant of any lower-bound superharmonic function v. Given two such functions v_1 and v_2, and given α and $\beta \in R^+$, we have $\boldsymbol{H}(\alpha v_1+\beta v_2)=\alpha\boldsymbol{H}(v_1)+\beta\boldsymbol{H}(v_2)$. (See [27, p. 284].)

Definition. Given $\bar{W}^Q$, the harmonic part of $\bar{W}^Q - W$ is

$$\Gamma_Q = \bigcap_{p\in P_W}\{z\in \bar{W}^Q - W : \liminf_W p(z)=0\}.$$

Remark. Any compactification of W is $\bar{W}^Q$ where $Q=\{f\mid W: f\in C(\bar{W})\}$.

Proposition 5.1. *If v is a lower-bounded superharmonic function on W with $\liminf_W v(z)\geqq 0$ at each $z\in\Gamma_Q$ for some family Q, then $v\geqq \mathbf{0}$ in W.*

Proof. Given $\varepsilon>0$, the set

$$A_\varepsilon = \{z\in\bar{W}^Q - W : \liminf_W v(z)\leqq -\varepsilon\}$$

is compact in $(\bar{W}^Q - W)-\Gamma_Q$. There is a finite sum of potentials $p = p_1+\cdots+p_m$ which is also a potential such that $\liminf_W p(z)\geqq\delta$ for some $\delta>0$ and every $z\in A_\varepsilon$. Thus for some $\alpha>0$, $\liminf_W(v+\alpha p+\boldsymbol{\varepsilon})(z)\geqq 0$ for every $z\in\bar{W}^Q - W$, whence $v+\alpha p+\boldsymbol{\varepsilon}\geqq\mathbf{0}$ on W since $v+\alpha p+\boldsymbol{\varepsilon}$ takes a minimum value on $\bar{W}^Q$ and a nonconstant function in $\mathcal{H}_W$ cannot take a nonpositive minimum in W. Therefore,

$$v+\alpha p+\boldsymbol{\varepsilon}\geqq\boldsymbol{H}(v+\alpha p+\boldsymbol{\varepsilon})=\boldsymbol{H}(v)+\boldsymbol{H}(\boldsymbol{\varepsilon})\geqq 0,$$

so $v+\boldsymbol{\varepsilon}\geqq\boldsymbol{H}(v)+\boldsymbol{H}(\boldsymbol{\varepsilon})\geqq 0$, i.e., $v\geqq-\boldsymbol{\varepsilon}$ for every $\varepsilon>0$. Thus $v\geqq\mathbf{0}$ on W. ∎

Let DP_W be the set of bounded continuous functions f on W for which there exists a potential p_f with $|f|\leqq p_f$ on W. We now consider compactifications $\bar{W}^Q$ where $B\mathcal{H}_W\subset Q\subset B\mathcal{H}_W\cup DP_W$; $\bar{W}^{B\mathcal{H}_W}$ is the Feller compactification of W, and $\bar{W}^{(B\mathcal{H}_W\cup DP_W)}$ is the Wiener compactification of W. Let CP_W denote the set of bounded continuous potentials on W. In Section 6, we shall work with $\bar{W}^{(B\mathcal{H}_W\cup CP_W)}$.

Proposition 5.2. *If $B\mathcal{H}_W\subset Q\subset B\mathcal{H}_W\cup DP_W$, then the restriction mapping $\rho: h\to h\mid\Gamma_Q$ is an isometric isomorphism from the Banach lattice $B\mathcal{H}_W$ onto $C(\Gamma_Q)$. The mapping ρ preserves the lattice operations on $B\mathcal{H}_W$ and $\rho(\boldsymbol{H}(\mathbf{1}))=\mathbf{1}$.*

PROOF (from [27]). That ρ is an isometry into $C(\Gamma_Q)$ follows from Proposition 5.1 and the fact that for any set A and any real-valued function f, $\sup_{x\in A}|f(x)|\leqq\alpha$ if and only if $\boldsymbol{\alpha}+f\geqq\mathbf{0}$ and $\boldsymbol{\alpha}-f\geqq\mathbf{0}$ on A. Identifying functions with their extensions, it is clear that $B\mathcal{H}_W$ separates the points of Γ_Q. Since $\mathbf{1}-\boldsymbol{H}(\mathbf{1})$ is a potential, $\mathbf{1}=\boldsymbol{H}(\mathbf{1})$ on Γ_Q. Similarly $f\wedge g=f\wedge_{*}g$ on Γ_Q for each pair $f,g\in B\mathcal{H}_W$. Since $\rho[B\mathcal{H}_W]$ is closed, ρ is surjective by the lattice form of the Stone–Weierstrass Theorem. ∎

Note that for all cases of Proposition 5.2, the sets Γ_Q are equivalent up to homeomorphisms. (See [27, p. 294].) The next result is usually stated for the Wiener compactification.

PROPOSITION 5.3. *If $B\mathcal{H}_W\cup CP_W\subset Q\subset B\mathcal{H}_W\cup DP_W$, then every positive harmonic function has a continuous extension to $\bar{W}^Q$.*

PROOF. Fix $h\geqq\mathbf{0}$ in $\mathcal{H}_W$. For each $n\in N$, $\boldsymbol{n}\wedge h$ is the sum of a potential and harmonic function and has a continuous extension to $\bar{W}^Q$. The upper envelope $\hat{h}$ of the family $\{\boldsymbol{n}\wedge h : n\in N\}$ is lower semicontinuous on $\bar{W}^Q$, and so $\hat{h}$ can only be discontinuous at a point $z\in\bar{W}^Q-W$ with $\hat{h}(z)=\alpha<+\infty$. But in this case, for $n>\alpha$ we have $\boldsymbol{n}\wedge h<\boldsymbol{n}$ in some neighborhood U of z, whence $\boldsymbol{n}\wedge h=\hat{h}$ in U. Thus $\hat{h}$ is a continuous extension of h on $\bar{W}^Q$. ∎

Finally we recall the following fact about quotients of the Wiener compactification.

PROPOSITION 5.4. *If $\bar{W}=W^Q$ for $B\mathcal{H}_W\subset Q\subset B\mathcal{H}_W\cup DP$ and $\tilde{W}$ is a compactification of W with $\bar{W}\geqq\tilde{W}$, then $\tilde{W}-W$ is a resolutive boundary for W. Let μ_x and $\tilde{\mu}_x$ denote harmonic measure with respect to x on $\bar{W}-W$ and $\tilde{W}-W$ respectively, and let φ be the continuous map from $\bar{W}$ onto $\tilde{W}$ such that $\varphi(x)=x$ for each $x\in W$. Then for each $x\in W$ and $f\in C(\tilde{W}-W)$ we have*

$$\int_{\tilde{W}-W} f\,d\tilde{\mu}_x=\int_{\bar{W}-W} f\circ\varphi\,d\mu_x .$$

PROOF. Let Γ and $\tilde{\Gamma}$ be the harmonic parts of $\bar{W}-W$ and $\tilde{W}-W$ respectively. (Then $\varphi[\Gamma]=\tilde{\Gamma}$, see [30, theor. 3.1.1].) Let f be continuous on $\tilde{W}-W$ and let h be the harmonic extension of $f\circ\varphi$ on Γ. Fix $\varepsilon>0$ in R. For each point $z\in\tilde{W}-W$ and each neighborhood U of $\varphi^{-1}[z]$ there is a neighborhood U' of z such that $\varphi^{-1}[U']\subset U$ ([30, p. 57]). Since points of Γ are regular, there is a potential p such that

$$\liminf_{W}(p+h+\tfrac{1}{2}\boldsymbol{\varepsilon})(y)\geqq f\circ\varphi(y)$$

for each $y\in\bar{W}-W$, and so

$$\liminf_{W} (p + h + \varepsilon)(z) \geqq f(z)$$

for each $z \in \tilde{W} - W$. It follows that for $\tilde{W}$,

$$\bar{H}(f, W) \leqq \boldsymbol{H}(p + h + \varepsilon) = \boldsymbol{H}(p) + \boldsymbol{H}(h) + \boldsymbol{H}(\varepsilon) = h + \boldsymbol{H}(\varepsilon)$$
$$\leqq h + \varepsilon.$$

Since ε and f are arbitrary, $\bar{H}(f, W) \leqq h$ and $\bar{H}(-f, W) \leqq -h$, i.e.,

$$\underline{H}(f, W) = -\bar{H}(-f, W) \geqq h.$$

Thus, for each $x \in W$,

$$\int_{\bar{W}-W} f \circ \varphi d\mu_x = h(x) = \underline{H}(f, W)(x) = \bar{H}(f, W)(x) = \int_{\tilde{W}-W} f d\tilde{\mu}_x. \quad \blacksquare$$

COROLLARY 5.5. *Given $\bar{W}$ and $\tilde{W}$ as above, $\tilde{\Gamma}$ is the support of $\tilde{\mu}_x$ for each $x \in W$, and if $z \in \tilde{W} - W$ is regular, then $z \in \tilde{\Gamma}$.*

PROOF. Since Γ is compact in $\bar{W} - W$ and is the support of μ_x for each $x \in W$, only points of Γ are regular for $\bar{W} - W$. Since $\varphi[\Gamma] = \tilde{\Gamma}$, the corollary now follows from the theorem. ∎

If $B\mathcal{H}_W \subset Q \subset B\mathcal{H}_W \cup DP_W$, then a point of $\bar{W}^Q - W$ is regular if and only if it is in Γ_Q If $z \in \Gamma_Q$ then by Theorem 3.2 the extension of every standard neighborhood of z intersects the boundary of every internal inner region containing all standard compact sets. We now give a new result due to A. Cornea and the author showing that this property characterizes the points of Γ_Q if $\bar{W}^Q$ is the Wiener compactification of W.

THEOREM 5.6 (A. Cornea, P. A. Loeb). *Let $\bar{W}$ be the Wiener compatification of W and fix $z \in \bar{W} - W$. Then z is a point of Γ, the harmonic part of $\bar{W} - W$, if and only if for each open neighborhood U of z and each countable exhaustion $\{\Omega_n\}$ of W by inner regions, there is an $n_0 \in N$ so that for all $n \geqq n_0$, $\partial\Omega_n \cap U \neq \emptyset$. Thus if $z \notin \Gamma$ there is an open neighborhood U of z and an internal inner region Ω with $\partial\Omega \subset m(\infty)$ such that ${}^*U \cap \partial\Omega = \emptyset$.*

PROOF. Assume $z \notin \Gamma$ and let U be an open neighborhood of z with $\bar{U} \cap \Gamma = \emptyset$. Let U_1 be an open neighborhood of z with $U_1 \subset \bar{U}_1 \subset U$. There is a potential p_1 on W with lim inf $p_1 \geqq 2$ at each point $x \in \bar{U}_1 - W$. The set

$$K = \{x \in W : p_1 \leqq 1\} \cap \bar{U}_1$$

is compact in W, and so there is a potential p_2 on W with $p_2 \geqq 1$ on K. Therefore, any function on W with values between 0 and 1 which vanishes on $W - U_1$ is dominated by the potential $p_1 + p_2$ and hence has a continuous extension to $\bar{W}$. Let $\{\Omega_n\}$ be any countable exhaustion of W by inner regions, and let U_2 be an open neighborhood of z with $U_2 \subset \bar{U}_2 \subset U_1$ such that $U_2 \cap \partial\Omega_n \neq \emptyset$ for all $n \geqq n_0$ for some $n_0 \in N$. (If no such U_2 exists, we are done.) Let $f = \mathbf{0}$ on $\partial\Omega_{n_0+j} \cap \bar{U}_2$ for all even j, and let $f = \mathbf{1}$ on $\partial\Omega_{n_0+j} \cap \bar{U}_2$ for all odd j. Extend f continuously to W so that $\mathbf{0} \leqq f \leqq \mathbf{1}$ and $f \mid W - U_1 = \mathbf{0}$. It now follows that f has a continuous extension to all of $\bar{W}$ and in particular to the point z. Therefore, there is a neighborhood $U_3 \subset U_2$ of z such that $U_3 \cap \partial\Omega_{n_0+j} = \emptyset$ either for all even j or for all odd j. In any case, there is a $j \in {}^*N - N$ such that ${}^*U_3 \cap \partial\Omega_{n_0+j} = \emptyset$. The rest follows from Theorem 3.2. ∎

6. An almost everywhere regular boundary supporting the maximal representing measures for bounded harmonic functions

In this section we establish the existence of an ideal boundary Δ for W such that the points of Δ correspond to non-negative harmonic functions, Δ supports the maximal representing measures for positive bounded and quasibounded harmonic functions, and almost all points of Δ are regular for the Dirichlet problem. The results of this section were first developed as an extension of Section 4 using nonstandard analysis throughout. Many of the constructions and arguments have been made "standard" as the section has been refined, but the original intuition should be acknowledged.

Recall that we are assuming the existence on W of a positive potential and a bounded harmonic function not identically equal to 0. Let Q be the set of bounded harmonic functions $B\mathcal{H}_W$ together with the continuous bounded potentials on W. We shall work with the compactification $\bar{W}^Q$ which we denote by $\bar{W}$; we let $\Gamma = \Gamma_Q$. The reader can, if he or she chooses, assume that $\bar{W}$ is the Wiener compactification; our choice of $\bar{W}$ is the smallest compactification yielding the desired results.

Fix a point $x_0 \in W$. For each $x \in W$, let μ_x denote harmonic measure on $\bar{W} - W$; the support of μ_x is Γ. Given $x \in W$, the Radon–Nikodym derivative $d\mu_x / d\mu_{x_0}$ is bounded and bounded away from 0 on Γ; the bounds can easily be obtained as the constants for Harnack's inequality associated with W and the compact set $\{x_0, x\}$. Moreover, if h is the upper envelope of any family $\{h_\alpha\}$ directed by increasing order in $B\mathcal{H}_W$, and if h is bounded, then $h \in \mathcal{H}$ by Axiom III, and

$$h(x_0) = \sup_\alpha h_\alpha(x_0) = \sup_\alpha \int_\Gamma h_\alpha \, d\mu_{x_0} \leqq \int_\Gamma h \, d\mu_{x_0} = h(x_0).$$

It follows (see remark 2 on page 293 of [27]) that $d\mu_x/d\mu_{x_0}$ is equal to a continuous function on Γ, μ_{x_0}-almost everywhere. We may therefore assume that $d\mu_x/d\mu_{x_0}$ is the trace on Γ of a bounded harmonic function of y, $r(x, y)$. That is, we set

$$r(x,y)=\int_\Gamma \frac{d\mu_x}{d\mu_{x_0}}\frac{d\mu_y}{d\mu_{x_0}}\,d\mu_{x_0}$$

for each pair $x, y \in W$. Given $x, y \in W$, we may assume that $r(x,\cdot)$ and $r(\cdot, y)$ are continuous on $\bar{W}$; clearly $r(x,y)=r(y,x)$.

Let $\tilde{Q}=\{r(x,\cdot)\in B\mathscr{H}_W : x\in W\}$, and let $\tilde{W}=\bar{W}^{\tilde{Q}}$. Let φ be the continuous mapping of $\bar{W}$ onto $\tilde{W}$ such that for $x\in W$, $\varphi(x)=x$, and for $z_1, z_2 \in \bar{W}-W$, $\varphi(z_1)=\varphi(z_2)$ if and only if $r(x,z_1)=r(x,z_2)$ for each $x\in W$. The compactification $\tilde{W}$ is the type of compactification considered by Thomas E. Armstrong in chapter 11 of [2]. The results in [2] have little relation, however, to those we shall now obtain, except that Proposition 6.1 can be obtained as a corollary of results in [2]. (In an unpublished manuscript, Stuart P. Lloyd has considered similar quotients of just the harmonic part of the Feller boundary from a probabilistic viewpoint.) Note that the compactification $\tilde{W}$ does not depend on the choice of x_0 since $d\mu_x/d\mu_{x_1}=d\mu_x/d\mu_{x_0}\,|\,d\mu_{x_1}/d\mu_{x_0}$ for any point $x_1\in W$.

Given $x\in W$, we let $q(x,\cdot)$ denote the continuous extension of $r(x,\cdot)$ to $\tilde{W}$ and $q(\cdot,x)$ denote the continuous extension of $r(\cdot,x)$ to $\tilde{W}$. If $x\in W$, $z\in\tilde{W}$, and $z'\in\bar{W}$ with $\varphi(z')=z$, then

$$q(x,z)=r(x,z')=\lim_{y\in W, y\to z'} r(x,y)=\lim_{y\in W, y\to z'} r(y,x)=r(z',x)=q(z,x).$$

Let $\Delta=\tilde{W}-W$, and let $\tilde{\Gamma}$ be the harmonic part of Δ. Given $x\in W$, let $\tilde{\mu}_x$ denote harmonic measure on Δ with respect to x; by Corollary 5.5, the support of $\tilde{\mu}_x$ is $\tilde{\Gamma}$. Let

$$\Phi_{x_0}=\{h\in\mathscr{H}_W : h>0 \quad\text{and}\quad h(x_0)=1\},$$

and let

$$\Phi'_{x_0}=\{h\in\mathscr{H}_W : h\geqq 0 \quad\text{and}\quad h(x_0)\leqq 1\}.$$

Both Φ_{x_0} and Φ'_{x_0} are compact in the u.c.c. topology by Theorem 2.1. We now show that the function $q(x,y)$ on $W\times\tilde{W}$ has many of the properties of the Poisson kernel for the unit disc.

PROPOSITION 6.1. *The function $q(x,y)$ on $W\times\tilde{W}$ has the following properties:*

i) *if $x\in W$, $q(x,\cdot)$ is continuous on $\tilde{W}$, harmonic on W, and $0<q(x,x_0)\leqq 1$. Moreover, $q(x,\cdot)|\Delta$ represents the Radon–Nikodym derivative $d\tilde{\mu}_x/d\tilde{\mu}_{x_0}$;*

ii) *if $z\in\Delta$, $q(\cdot,z)$ is harmonic on W, and $q(x_0,z)\leqq 1$;*

iii) *the mapping* $T: x \to q(\cdot, x)$ *from* $\tilde{W}$ *into* Φ'_{x_0} *is continuous with respect to the u.c.c. topology on* Φ'_{x_0}.
If $z_1, z_2 \in \Delta$ *and* $z_1 \neq z_2$, *then* $T(z_1) \neq T(z_2)$. *Thus* $T \mid \Delta$ *is a homeomorphism from* Δ *onto a subset of* Φ'_{x_0}, *and so* Δ *is a metric space*;

iv) *if* $z \in \tilde{\Gamma}$, $q(x_0, z) = 1$. *Thus* $T \mid \tilde{\Gamma}$ *is a homeomorphism from* $\tilde{\Gamma}$ *onto a subset of* Φ_{x_0}. *If* $\mathbf{1}$ *is harmonic*, $q(x, x_0) = 1$ *for all* $x \in \tilde{W}$.

PROOF. To prove (i), fix $x \in W$. By definition $q(x. \cdot)$ is continuous on $\tilde{W}$ and harmonic on W. Moreover,

$$q(x, x_0) = r(x, x_0) = \int_\Gamma \frac{d\mu_x}{d\mu_{x_0}} \, d\mu_{x_0} = \int_\Gamma d\mu_x = \boldsymbol{H}(\mathbf{1})(x),$$

and $0 < \boldsymbol{H}(\mathbf{1})(x) \leqq \mathbf{1}$. By Proposition 5.4, if $f \in C(\Delta)$ and $x \in W$, then

$$\begin{aligned}\int_\Delta f(y) q(x, y) \, d\tilde{\mu}_{x_0}(y) &= \int_\Gamma (f \circ \varphi(z)) r(x, z) \, d\mu_{x_0}(z) \\ &= \int_\Gamma f \circ \varphi(z) \, d\mu_x(z) = \int_\Delta f(y) \, d\tilde{\mu}_x(y).\end{aligned}$$

It follows that $q(x, \cdot)$ is a representative of $d\tilde{\mu}_x / d\tilde{\mu}_{x_0}$ on Δ.

To prove (ii) and (iii), let T_W be the restriction to W of T. Given a finite set $x, y_1, \cdots, y_n$ in W and $\varepsilon > 0$ in R, there is a neighborhood U of x such that if $x' \in U$ and $1 \leqq i \leqq n$, then $|q(x', y_i) - q(x, y_i)| < \varepsilon$. It follows that T_W is continuous with respect to the topology of pointwise convergence on Φ'_{x_0}, which is the same as the u.c.c. topology on Φ'_{x_0} by Theorem 2.1. Since Φ'_{x_0} is compact, we can imbed W in a unique (up to homeomorphism) compactification $\hat{W}$ such that T_W has a continuous extension to $\hat{W}$, $T_W: \hat{W} \to \Phi'_{x_0}$, and T_W separates the points of $\hat{W} - W$. To do this, we use the method of [23] or the analogue of the methods described in Section 5. For each $z \in \hat{W}$, let $\hat{q}(z, \cdot) = T_W(z)$. If $z \in W$, $\hat{q}(z, \cdot) = q(z, \cdot) = r(z, \cdot)$. Given $z \in \hat{W} - W$, $\varepsilon > 0$ in R and $x \in W$, $\{x\}$ is compact, so there is a neighborhood U of z in $\hat{W}$ with the property that for any $z' \in U$, $|\hat{q}(z', x) - \hat{q}(z, x)| < \varepsilon$. Therefore, $\hat{q}(\cdot, x)$ is a continuous extension to $\hat{W}$ of $r(\cdot, x)$ on W. Moreover, the functions $\{\hat{q}(\cdot, x): x \in W\}$ separate the points of $\hat{W} - W$. By the definition of $\tilde{W}$, there is a homeomorphism ψ from $\tilde{W}$ onto $\hat{W}$ with $\psi(x) = x$ for each $x \in W$, and for each $z \in \tilde{W}$, $q(z, \cdot) = \hat{q}(\psi(z), \cdot)$. Clearly (ii) and (iii) now follow.

To prove (iv) note that for each $x \in W$,

$$q(x, x_0) = r(x, x_0) = \int_\Gamma d\mu_x = \boldsymbol{H}(\mathbf{1})(x).$$

Since $\boldsymbol{H}(\mathbf{1})$ is equal to $\mathbf{1}$ on Γ, $r(z, x_0)$ equals 1 for each $z \in \Gamma$. If $y \in \tilde{\Gamma}$, there is a $z \in \Gamma$ with $\varphi(z) = y$ (see [30, theorem 3.1.1]), so

$$q(x_0, y) = q(y, x_0) = r(z, x_0) = 1 .$$

If $\mathbf{1} \in \mathcal{H}_W$, $\boldsymbol{H}(\mathbf{1}) = \mathbf{1}$, so $q(\cdot, x_0) = \mathbf{1}$. ∎

We next establish an important difference between Δ and the boundary of R. S. Martin [29]. M. G. Shur has shown in [35] that there can exist a minimal irregular point z on the Martin boundary such that $\{z\}$ has positive harmonic measure. It will follow from the next result and Theorem 6.4 that this cannot be the case for Δ.

DEFINITION. A point $z \in \Delta$ is called minimal if $q(z, \cdot)$ is minimal.

THEOREM 6.2. *A minimal point $z \in \Delta$ is regular if $q(z, x_0) = 1$.*

PROOF. We give a nonstandard and a standard proof. For each $x \in W$, $\tilde{\mu}_x(\tilde{\Gamma}) \leqq 1$, so $\{\tilde{\mu}_x : x \in W\}$ is compact in the *weak topology for Borel measures on $\tilde{\Gamma}$. Let δ_z denote the probability measure on Δ such that $\delta_z(\Delta - \{z\}) = 0$. Then z is a regular point if and only if $\lim_{x \in W, x \to z} \tilde{\mu}_x = \delta_z$ in the *weak topology. This is the case if for each nonstandard $y \in m(z)$, δ_z is the standard part ${}^\circ\tilde{\mu}_y$ of $\tilde{\mu}_y$ with respect to the *weak topology. But for each standard $w \in W$ we have

$$\int_{\tilde{\Gamma}} {}^*q(s, w)\, d\tilde{\mu}_y(s) = {}^*q(y, w) \simeq q(z, w) .$$

Therefore,

$$\int_{\tilde{\Gamma}} q(s, w)\, d^\circ\tilde{\mu}_y(s) = q(z, w)$$

for each $w \in W$, and letting $w = x_0$ we have $\int_{\tilde{\Gamma}} d^\circ\tilde{\mu}_y = 1$. Since ${}^\circ\tilde{\mu}_y$ is a probability measure on Φ_{x_0} and ${}^\circ\tilde{\mu}_y$ represents the extreme element $q(z, \cdot)$ in Φ_{x_0}, ${}^\circ\tilde{u}_y = \delta_z$. (See [32, p. 8].)

A standard version of this proof is obtained by taking an arbitrary net x_α converging to z in W and letting ν be a *weak cluster point of the net $\tilde{\mu}_{x_\alpha}$. We must show that $\nu = \delta_z$. There is a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ such that for each $w \in W$,

$$\int_{\tilde{\Gamma}} q(s, w)\, d\nu(s) = \lim_\beta \int_{\tilde{\Gamma}} q(s, w)\, d\tilde{\mu}_{x_\beta}(s)$$

$$= \lim_\beta q(x_\beta, w) = q(z, w),$$

and $\int_{\tilde{\Gamma}} d\nu = q(z, x_0) = 1$. Since ν is a probability measure on Φ_{x_0} and ν represents $q(z. \cdot)$, $\nu = \delta_z$. ∎

EXAMPLE. We show the assumption in Theorem 6.2 that $q(z, x_0) = 1$ cannot be omitted. Consider the space $W = \{x \in R : 0 < x < +\infty\}$ with $\mathcal{H}_\Omega$ consisting of all linear combinations of the functions e^x and $1+e^{-x}$ on Ω for each open interval $\Omega \subset W$. Here, $2-(1+e^{-x})=1-e^{-x}$ is a potential on W, and $\{0\}$ is the harmonic part of $\bar{W}$. Let $x_0 = 1$. Then $\mu_x(\{0\}) = \frac{1}{2}(1+e^{-x})$ for each $x \in W$ and so $\mu_{x_0}(\{0\}) = \frac{1}{2}(1+e^{-1})$. Given $x, y \in W$,

$$r(x,y) = \int_\Gamma \frac{d\mu_x}{d\mu_{x_0}} \frac{d\mu_y}{d\mu_{x_0}} d\mu_{x_0} = \tfrac{1}{2}(1+e^{-1})\left(\frac{1+e^{-x}}{1+e^{-1}}\right)\left(\frac{1+e^{-y}}{1+e^{-1}}\right),$$

and $r(0,x) = (1+e^{-x})/(1+e^{-1})$ while

$$r(+\infty, x) = \tfrac{1}{2}\frac{1+e^{-x}}{1+e^{-1}} = \tfrac{1}{2} r(0,x).$$

Therefore, $\Delta = \{0, +\infty\}$ and $q(+\infty, \cdot)$ is minimal, but $+\infty$ is not a regular point of Δ.

Recall that an unbounded, positive harmonic function h on W is called quasibounded if it is the limit of an increasing sequence $\{h_n\}$ of bounded harmonic functions. Let h and h_n also denote the continuous extensions of these functions to W. Given $w \in W$, we have

$$h(w) = \lim_{n\to\infty} h_n(w) = \lim_{n\to\infty} \int_\Gamma h_n d\mu_w \leqq \int_\Gamma h d\mu_w$$

$$= \lim_{m\to\infty} \int_\Gamma (h \wedge m) d\mu_w = \lim_{m\to\infty} \boldsymbol{H}(h \wedge \boldsymbol{m})(w) \leqq h(w).$$

Therefore, for each $w \in W$,

$$h(w) = \int_\Gamma h\, d\mu_w = \int_\Gamma h(y) r(y,w)\, d\mu_{x_0}(y).$$

DEFINITION. For each $h \in \mathscr{H}^+_W$, let ν_h be the measure on $\tilde{\Gamma}$ defined by setting

$$\nu_h(A) = \int_{\Gamma \cap \varphi^{-1}[A]} h\, d\mu_{x_0}$$

for each Borel set $A \subset \tilde{\Gamma}$.

THEOREM 6.3. *If $h \in \Phi_{x_0}$ is bounded or quasibounded, then ν_h considered now as a probability measure on the set $\{q(z,\cdot): z \in \tilde{\Gamma}\} \subset \Phi_{x_0}$ is the maximal representing measure for h.*

PROOF. For each $w \in W$,

$$h(w) = \int_\Gamma r(y,w) h(y)\, d\mu_{x_0}(y) = \int_{\tilde{\Gamma}} q(z,w)\, d\nu_h(z).$$

That is, ν_h represents h on Φ_{x_0}. The mapping $h \to \nu_h$ is affine, so, by Corollary 4.4, ν_h is maximal if h is bounded. If h is quasibounded and A is a Borel subset of $\tilde{\Gamma}$, then

$$\nu_h(A) = \lim_{m\to\infty} \int_{\Gamma\cap\varphi^{-1}[A]} (h \wedge m)\, d\mu_{x_0} = \lim_{m\to\infty} \nu_{H(h\wedge m)}(A).$$

For each $m \in N$, $\nu_{H(h\wedge m)}$ is supported by extreme points $\mathscr{E}_{x_0}$ of Φ_{x_0}, so ν_h is supported by $\mathscr{E}_{x_0}$. Therefore ν_h is maximal. ∎

THEOREM 6.4. *Almost all points of Δ with respect to harmonic measure are minimal points in $\tilde{\Gamma}$ and are therefore regular.*

PROOF. If A is a Borel set in $\tilde{\Gamma}$, then it follows from Proposition 5.4 that $\tilde{\mu}_{x_0}(A) = \mu_{x_0}(\varphi^{-1}[A])$. Therefore

$$\nu_{H(1)}(A) = \int_{\Gamma\cap\varphi^{-1}[A]} H(1)\, d\mu_{x_0} = \int_{\Gamma\cap\varphi^{-1}[A]} d\mu_{x_0} = \int_A d\tilde{\mu}_{x_0} = \tilde{\mu}_{x_0}(A)$$

for each Borel set $A \subset \tilde{\Gamma}$. That is, $\nu_{H(1)} = \mu_{x_0}$. Let $c = H(1)(x_0)$. Then $\mu_{x_0} = \nu_{H(1)} = c\nu_{c^{-1}H(1)}$ is supported by the minimal point of $\tilde{\Gamma}$. Thus harmonic measure for each $x \in W$ is supported by the minimal point of $\tilde{\Gamma}$, and these are all regular points by Theorem 6.2. ∎

When it exists, the Martin compactifications of W may be quite unlike the compactification $\tilde{W}$. For example, given $\mathscr{H}$ and a positive $h \in \mathscr{H}_W$, the class $\mathscr{H}/h = \{f/h : f \in \mathscr{H}\}$ is a harmonic class (see [4], [5], or [22]) with the same Martin boundary for W as $\mathscr{H}$. If h is minimal, however, the only bounded elements of $\mathscr{H}/h$ are multiples of $\mathbf{1}$. In this case, Γ is a single point and $\tilde{W}$ is the one point compactification of W.

We consider now under what circumstances Δ is the support of the maximal representing measure for every $h \in \Phi_{x_0}$, that is, under what circumstances the family $\{q(z,\cdot): z \in \tilde{\Gamma}\} \supset \mathscr{E}_{x_0}$. We also give a criterion for $\tilde{W} - W$ to be a given boundary for W, e.g., the topological boundary. An application of this criterion to the open unit disc $D = \{z \in C : |z| < 1\}$ shows that $\tilde{D}$ is the closed disc $\{z \in C : |z| \leqq 1\}$ which is the Martin compactification of D.

Note that if Ω is an internal disc $\{z \in {}^*C : |z| < M\}$ contained in the extension ${}^*D$ of the unit disc D, with $M < 1$ and $M \simeq 1$, then for each $y \in \partial\Omega$ there is an $\alpha \in {}^*[0, 2\pi]$ such that $y = Me^{i\alpha}$ and ${}^\circ y = e^{i{}^\circ\alpha}$. The value of the extension of the Poisson kernel at each point $y \in \partial\Omega$ and $z = re^{i\theta} \in D$ is

$$\frac{M^2 - r^2}{M^2 - 2Mr\cos(\theta - \alpha) + r^2} \simeq \frac{1 - r^2}{1 - 2r\cos(\theta - {}^\circ\alpha) + r^2}.$$

Here, the right side is the value of the standard Poisson kernel at $°y$ and z. These facts suggest the following general result.

THEOREM 6.5. *Let $\hat{W}$ be a resolutive compactification of W. Assume that for each $x \in W$ there is a continuous real-valued function $p(\cdot, x)$ on $\hat{W} - W$ such that for each $z \in \hat{W} - W$, $p(z, \cdot) \in \Phi_{x_0}$ and if z_1 and z_2 are distinct in $\hat{W} - W$, $p(z_1, \cdot) \neq p(z_2, \cdot)$. Then the mapping $z \to p(z, \cdot)$ is a homeomorphism from $\hat{W} - W$ onto a subset of Φ_{x_0}. Also assume that there is an internal inner region $\Omega \subset {}^*W$ with $\partial\Omega \subset m(\infty)$ such that for each standard $x \in W$ there is an internal representative $\pi(\cdot, x)$ of the Radon–Nikodym derivative $d\mu_x^\Omega / d\mu_{x_0}^\Omega$ on $\partial\Omega$ with $°\pi(y, x) = p(S(y), x)$ for each $y \in \partial\Omega$, where $S(y)$ is the standard part of y in $\hat{W} - W$. For each $h \in \Phi_{x_0}$, let σ_h be the internal measure on the collection $\mathscr{A}$ of internal Borel subsets of $\partial\Omega$ such that for each $A \in \mathscr{A}$,*

$$\sigma_h(A) = \int_A {}^*h \, d\mu_{x_0}^\Omega .$$

Let $°\sigma_h$ be the unique extension of σ_h to the smallest (external) σ-algebra $\mathscr{M}$ in $\partial\Omega$ with $\mathscr{M} \supset \mathscr{A}$. (See [25].) *Then the mapping S is measurable with respect to the Borel sets in $\hat{W} - W$ and the σ-algebra $\mathscr{M}$, and if for $h \in \Phi_{x_0}$ and each Borel set $B \subset \hat{W} - W$ we set*

$$\rho_h(B) = °\sigma_h(S^{-1}[B]),$$

then ρ_h is the maximal representing measure for h on Φ_{x_0}. Moreover, $\hat{W}$ and $\tilde{W}$ are equivalent compactifications of W (i.e., $\hat{W} \geqq \tilde{W}$ and $\tilde{W} \geqq \hat{W}$) if for each $x \in W$, $p(\cdot, x)$ is the restriction from $\hat{W}$ to $\hat{W} - W$ of a continuous function which is harmonic on W. In this case, $p(\cdot, x) = q(\cdot, x)$ for each $x \in W$, and the set $\{q(z, \cdot) : z \in \Delta\} \supset \mathscr{E}_{x_0}$.

PROOF. As in the proof of Proposition 6.1, the one-one mapping $z \to p(z, \cdot)$ is continuous with respect to the topology of point-wise convergence which is the u.c.c. topology on Φ_{x_0}, so the mapping is a homeomorphism.

Let K be a compact subset of $\hat{W} - W$ and let $\{U_n\}$ be a decreasing sequence of open neighborhoods of K with $\bigcap_{n=1}^\infty U_n = K$ such that for each $z \in (\hat{W} - W) - K$, there is an open neighborhood V of z and an $n \in N$ with $V \cap U_n = \emptyset$. Since $\hat{W} - W$ is a metric space and there is a countable exhaustion of W by compact sets, such a sequence exists. For each $n \in N$, ${}^*U_n \cap \partial\Omega \in \mathscr{A}$, and so $K' \equiv \bigcap_{n \in N}({}^*U_n \cap \partial\Omega) \in \mathscr{M}$. If $y \in \partial\Omega$ and $S(y) \in K$, then $y \in {}^*U_n \cap \partial\Omega$ for each $n \in N$, so $y \in K'$. If $y \in \partial\Omega$ and $S(y) \notin K$, Then there is a $n \in N$ and an open neighborhood V of $S(y)$ such that $V \cap U_n = \emptyset$. Since $y \in {}^*V$,

$y \notin {}^*U_n$, and so $y \notin K'$. Therefore, $S^{-1}[K] = K' \in \mathcal{M}$ for the arbitrary compact set $K \subset \hat{W} - W$, and so S is measurable.

Given $h \in \Phi_{x_0}$ and ρ_h on $\hat{W} - W$, it follows from theorem 3 of [25] that for each $x \in W$,

$$\int_{\hat{W}-W} p(z,x)\, d\rho_h(z) = \int_{\partial\Omega} {}^\circ\pi(y,x)\, d^\circ\sigma_h(y)$$

$$\simeq \int_{\partial\Omega} \pi(y,x)^* h(y) d\mu^{\Omega}_{x_0}(y) = h(x).$$

Since ρ_h thus represents h and the mapping $h \to \rho_h$ is affine, ρ_h is the maximal representing measure for h by Corollary 4.3.

Assume now that for each $x \in W$ we can extend $p(\cdot, x)$ so that $p(\cdot, x)$ is continuous on $\hat{W}$ and harmonic on W. Since the functions $\{p(\cdot, x): x \in W\}$ separate the points of $\hat{W} - W$, $\hat{W} \leqq \bar{W}$. Let $\hat{\varphi}$ be the continuous map from $\bar{W}$ onto $\hat{W}$ with $\hat{\varphi}(x) = x$ for each $x \in W$. For each $y \in \partial\Omega$, let ${}^\circ y$ denote the standard part of y in $\bar{W} - W$, and as before let $S(y)$ denote the standard part of y in $\hat{W} - W$. Given $y \in \partial\Omega$ and a standard open neighborhood U of $\hat{\varphi}({}^\circ y)$ in $\hat{W}$, since $\hat{\varphi}^{-1}[U]$ is a neighborhood of ${}^\circ y$, $y \in {}^*U$, and so $S(y) = \hat{\varphi}({}^\circ y)$. It follows that for each $x \in W$, $p(\hat{\varphi}({}^\circ y), x) = {}^\circ\pi(y, x)$. Let f be a continuous function on $\bar{W} - W$. Without loss of generality, we may assume that f is the restriction to $\bar{W} - W$ of a bounded $h \in \mathcal{H}_W$. Fix $x \in W$. For each $y \in \partial\Omega$,

$${}^*p(y,x) \simeq \pi(y,x).$$

Therefore, by Theorem 3.1

$$\int_{\Gamma} p(\hat{\varphi}(z),x) f(z) d\mu_{x_0}(z) \simeq \int_{\partial\Omega} {}^*p(y,x) h(y) d\mu^{\Omega}_{x_0}(y)$$

$$\simeq \int_{\partial\Omega} \pi(y,x)^* h(y)\, d\mu^{\Omega}_{x_0}(y) = h(x).$$

It follows that $p(\cdot, x)\circ\hat{\varphi}$ is a continuous representative of $d\mu_x / d\mu_{x_0}$ on $\bar{W} - W$, and so

$$p(\cdot, x)\circ\hat{\varphi} = r(\cdot, x)$$

on $\bar{W}$. Therefore, $\hat{W}$ and $\bar{W}$ are equivalent compactifications of W since they are both the Q-compactification for $Q = \{r(\cdot, x): x \in W\}$. The rest is clear. ∎

COROLLARY 6.6. *If W is the open disc $\{z \in C: |z| < 1\}$ and $\mathcal{H}$ is the family of harmonic functions on W in the usual sense, then $\bar{W} = \{z \in C: |z| \leqq 1\}$.*

For a more general result than Corollary 6.6, one can use Hunt and Wheeden's paper [19] which shows that the Martin compactification coincides with the Euclidean compactification for any bounded Lipshitz domain Ω in a Euclidean space. Moreover, every bounded harmonic function is the integral of an L_∞ function (with respect to harmonic measure) on $\partial\Omega$, all points of $\partial\Omega$ are regular, and given $x_0 \in \Omega$, for each $x \in \Omega$, $d\mu_x/d\mu_{x_0}$ is continuous on $\partial\Omega$. Since these Radon–Nikodym derivatives separate the points of $\partial\Omega$, it follows that the Euclidean closure $\bar{\Omega}$ is the compactification $\tilde{\Omega}$ considered here.

The use of the measurability of the standard part map in the proof of Theorem 6.5 occurred much later than its use in the proofs of Proposition 4.5 and Theorem 4.6. In the meantime, this device has been used by Robert M. Anderson in the article appearing in this volume for a construction of Wiener measure and a construction of Lebesgue measure on $[0,1]$.

A generalization is given of the radial limits considered in Fatou's Theorem by the notion of fine limits at points corresponding to minimal harmonic functions. The following definition of a fine limit is due to K. N. Gowrisankaran [16]; it was suggested by the classical concept of L. Naim [31].

Definition. Let h be a positive minimal harmonic function and let E be a set in W. The function R_h^E is the lower envelope of all positive superharmonic functions v on W such that $v \geqq h$ on E. The set E is called thin with respect to h if $R_h^E \lneqq h$ (in which case, $\mathbf{0}$ is the greatest non-negative harmonic function on W majorized by R_h^E). The fine filter $\mathcal{F}_h$ at h is the filter formed by the family of sets whose complements are thin with respect to h. A function on W has a fine limit at h if the limit with respect to the filter $\mathcal{F}_h$ exists. (See Brelot [6] for more details.)

We shall need the following application of a generalization of Fatou's theorem due to L. Naim and J. L. Doob in the classical case and K. Gowrisankaran [16] in the axiomatic framework employed here. (Gowrisankaran's assumption that W has a countable base for its topology is not needed for this result. Also see Armstrong [2, chap. 11].) Recall that $\tilde{\mu}_{x_0}$ is the representing measure for the greatest harmonic minorant of $\mathbf{1}$ on Φ_{x_0}.

Theorem 6.7 (Fatou–Naim–Doob–Gowrisankaran). *If $f \geqq 0$ is a $\tilde{\mu}_{x_0}$-integrable function on Δ and*

$$h(x) = \int_\Delta f(z)\, q(z,x)\, d\tilde{\mu}_{x_0}(z)$$

for each $x \in W$, then the fine limit of h exists and equals $f(z)$ for $\tilde{\mu}_{x_0}$-almost all $z \in \Delta$.

It is well known that the functions h for which Theorem 6.7 is applicable are the bounded and quasibounded functions in $\mathcal{H}_W^+$. Our construction of Δ yields a brief proof and an interesting consequence of this fact. The generalization of Theorem 6.8 to the differences of nonnegative harmonic functions is left to the reader.

THEOREM 6.8. *Let h be a nonnegative real-valued function on W.*

i) *If there is a Borel measurable function $f \geqq \mathbf{0}$ on Δ such that*

$$h(x) = \int_\Delta q(y, x) f(y)\, d\tilde{\mu}_{x_0}(y)$$

for each $x \in W$, then h is a bounded or quasibounded harmonic function on W with $h \leqq \boldsymbol{M}$ if $f \leqq \boldsymbol{M}$.

ii) *If h is a bounded or quasibounded harmonic function on W and ν_h is the maximal representing measure for h on Δ, then ν_h is absolutely continuous with respect to $\tilde{\mu}_{x_0}$ and, of course,*

$$h(x) = \int_\Delta q(y, x) \left(\frac{d\nu_h}{d\tilde{\mu}_{x_0}}(y) \right) d\tilde{\mu}_{x_0}(y).$$

iii) *Assume the hypothesis of* (ii) *hold and $f = d\nu_h / d\tilde{\mu}_{x_0}$ on Δ. There is a Borel set $B \subset \Gamma$ with $\mu_{x_0}(B) = 0$ so that if $z_1, z_2 \in \Gamma - B$ and $\varphi(z_1) = \varphi(z_2)$, then*

$$h(z_1) = h(z_2) = f \circ \varphi(z_1) = f \circ \varphi(z_2).$$

PROOF OF (i). Since for each $x \in W$,

$$h(x) = \int_\Delta q(y, x) f(y)\, d\tilde{\mu}_{x_0}(y) = \int_\Gamma r(z, x) f \circ \varphi(z)\, d\mu_{x_0}(z), \tag{1}$$

h is harmonic on W, and since

$$h(x) = \lim_{m \to \infty} \int_\Gamma r(z, x)\, (f \circ \varphi \wedge \boldsymbol{m})\, (z)\, d\mu_{x_0}(z),$$

h is either bounded or quasibounded. If $f \leqq \boldsymbol{M}$, then for each $x \in W$,

$$h(x) = \int_\Delta q(y, x) f(y)\, d\tilde{\mu}_{x_0}(y) \leqq M \int_\Delta q(y, x)\, d\tilde{\mu}_{x_0}(y)$$

$$= M \int_\Delta q(y, x)\, d\nu_{H(1)}(y) = M\boldsymbol{H}(\mathbf{1})(x) \leqq M.$$

PROOF OF (ii). If h is a bounded or quasibounded harmonic function and A is a Borel set in Δ with $\tilde{\mu}_{x_0}(A) = 0$, then by Theorem 6.3

$$\nu_h(A) = \int_{\Gamma \cap \varphi^{-1}[A]} h(z)\, d\mu_{x_0}(z) = 0$$

since

$$\int_{\Gamma \cap \varphi^{-1}[A]} d\mu_{x_0} = \int_A d\tilde{\mu}_{x_0} = 0 .$$

Proof of (iii). If h is a bounded harmonic function and $f = d\nu_h / d\tilde{\mu}_{x_0}$ on Δ, then Equation (1) holds. Let g be the continuous function on Γ such that $g = f \circ \varphi$ μ_{x_0}-almost everywhere on Γ. Then for each $x \in W$,

$$\int_\Gamma r(z,x)\, g(z)\, d\mu_{x_0}(z) = \int_\Gamma r(z,x)\, h(z)\, d\mu_{x_0}(z),$$

so $g = h \mid \Gamma$. Let $B = \{z \in \Gamma : h(z) \neq f \circ \varphi(z)\}$. Then $\mu_{x_0}(B) = 0$ and for each pair $z_1, z_2 \in \Gamma - B$ with $\varphi(z_1) = \varphi(z_2)$, we have

$$h(z_1) = f(\varphi(z_1)) = f(\varphi(z_2)) = h(z_2).$$

If h is quasibounded, let h_n be an increasing sequence in $B\mathcal{H}_W$ with limit h. For each $n \in N$ let $f_n = d\nu_{h_n} / d\tilde{\mu}_{x_0}$ on Δ and let B_n be the null set in Γ described above for h_n and f_n. Then $h(z) \geqq \sup_n h_n(z)$ for each $z \in \Gamma$, but

$$h(x_0) = \lim_{n \to \infty} \int_\Gamma h_n\, d\mu_{x_0} = \int_\Gamma \left(\sup_n h_n\right) d\mu_{x_0} \leqq \int_\Gamma h d\mu_{x_0} = h(x_0).$$

Therefore, if

$$B = \left\{ z \in \Gamma : h(z) \neq \sup_n h_n(z) \right\} \cup \bigcup_{m=1}^{\infty} B_m,$$

then $\mu_{x_0}(B) = 0$; for $z_1, z_2 \in \Gamma - B$ with $\varphi(z_1) = \varphi(z_2)$ we have

$$h(z_1) = \lim_{n \to \infty} h_n(z_1) = \lim_{n \to \infty} h_n(z_2) = h(z_2)$$

$$= \left(\sup_n f_n\right) \circ \varphi(z_1) = \left(\sup_n f_n\right) \circ \varphi(z_2). \qquad \blacksquare$$

Part (iii) of Theorem 6.8 was suggested by a result of T. Ikegami (lemma 2 in [20]) obtained for the Martin boundary. We now give a method of computing fine limits of bounded functions using an arbitrary internal inner region Ω with $\partial\Omega \subset m(\infty)$.

Definition. Given any $g \in \Phi_{x_0}$ and an internal inner region $\Omega \subset {}^*W$ with $\partial\Omega \subset m(\infty)$, let

$$\mathcal{G}(g,\Omega)=\left\{A\subset W:\int_{\partial\Omega\cup {}^*A} {}^*g\,d\mu^{\Omega}_{x_0}\simeq 1\right\}.$$

THEOREM 6.9. *For each* $g\in\Phi_{x_0}$ *and internal inner region* $\Omega\subset W$ *with* $\partial\Omega\subset M(\infty)$, $\mathcal{G}(g,\Omega)$ *is a filter in W. If g is extreme in* Φ_{x_0}, $\mathcal{G}(g,\Omega)$ *is a filter finer than the fine filter* $\mathcal{F}_g$ *at g. If f is a bounded real-valued function on W and f has a limit* α *with respect to* $\mathcal{G}(g,\Omega)$, *e.g., if the fine limit of f exists at g and is* α, *then*

$$\alpha={}^{\circ}\int_{\partial\Omega} {}^*f\,{}^*g\,d\mu^{\Omega}_{x_0}.$$

PROOF. Given $A,B\in\mathcal{G}(g,\Omega)$, since $\int_{\partial\Omega}{}^*g\,d\mu^{\Omega}_{x_0}=g(x_0)=1$, we have $\int_{\partial\Omega-A}{}^*g\,d\mu^{\Omega}_{x_0}\simeq 0$ and $\int_{\partial\Omega-B}{}^*g\,d\mu^{\Omega}_{x_0}\simeq 0$, whence $\int_{\partial\Omega\cap A\cap B}g^*\,d\mu^{\Omega}_{x_0}\simeq 1$. It follows that $\mathcal{G}(g,\Omega)$ is a filter. If $g\in\mathcal{E}_{x_0}$ and $W-B\in\mathcal{F}_g$, then for each $v\in\bar{\mathcal{H}}_W$ with $v\geqq g$ on B, we have ${}^*v\geqq{}^*g$ on ${}^*B\cap\partial\Omega$, and so for each $x\in W$

$$h(x)\equiv{}^{\circ}\int_{\partial\Omega\cap{}^*B} {}^*g\,d\mu^{\Omega}_{x}\leqq R^B_g(x).$$

Therefore the harmonic function $h=\mathbf{0}$, and so $W-B\in\mathcal{G}(g,\Omega)$. Thus $\mathcal{G}(g,\Omega)\supset\mathcal{F}_g$.

Let f be a bounded real-valued function on W, and assume that f has limit α with respect to $\mathcal{G}(g,\Omega)$. Then for any $\varepsilon>0$ in R,

$$A_\varepsilon=\{x\in W:\alpha-\varepsilon<f(x)<\alpha+\varepsilon\}\in\mathcal{G}(g,\Omega),$$

and so

$$\alpha-\varepsilon\simeq(\alpha-\varepsilon)\int_{\partial\Omega\cap{}^*A_\varepsilon}{}^*g\,d\mu^{\Omega}_{x_0}\leqq\int_{\partial\Omega\cap{}^*A_\varepsilon}{}^*g\,{}^*f\,d\mu^{\Omega}_{x_0}\simeq\int_{\partial\Omega}{}^*g\,{}^*f\,d\mu^{\Omega}_{x_0}$$

$$\simeq\int_{\partial\Omega\cap{}^*A_\varepsilon}{}^*g\,{}^*f\,d\mu^{\Omega}_{x_0}\leqq(\alpha+\varepsilon)\int_{\partial\Omega\cap{}^*A_\varepsilon}{}^*g\,d\mu^{\Omega}_{x_0}\simeq\alpha+\varepsilon.$$

Since ε is arbitrary in R^+,

$$\alpha={}^{\circ}\int_{\partial\Omega}{}^*f\,{}^*g\,d\mu^{\Omega}_{x_0}.$$

∎

COROLLARY 6.10. *If f is a bounded function with fine limit* α *at an extreme element* $g\in\Phi_{x_0}$, *and if* $\{\Omega_n\}$ *is an exhaustion of W by inner regions, then*

$$\alpha=\lim_{n\to\infty}\int_{\partial\Omega_n}fg\,d\mu^{\Omega_n}_{x_0}.$$

COROLLARY 6.11. *Fix an internal inner region* Ω *with* $\partial\Omega\subset m(\infty)$ *and an exhaustion* $\{\Omega_n\}$ *of W by standard inner regions. Given* $h\in B\mathcal{H}_W$, *let*

$$f_h(z) = {}^\circ\!\!\int_{\partial\Omega} {}^*h(y)^*q(z,y)\,d\mu_{x_0}^{\Omega}(y)$$

each $z \in \Delta$. *Then* f_h *represents* $d\nu_h / d\tilde{\mu}_{x_0}$, *the mapping* $h \to f_h$ *is linear, and*

$$f_h(z) = \lim_{n\to\infty} \int_{\partial\Omega_n} h(y) q(z,y)\,d\mu_{x_0}^{\Omega_n}(y)$$

for $\tilde{\mu}_{x_0}$*-almost all* $z \in \Delta$.

PROOF. See Theorems 6.7 and 6.9. ∎

In conclusion, we note that the existence of $\tilde{W}$ and the theory developed here have been established for a hyperbolic harmonic space W satisfying Axioms I–IV. Additional axioms, e.g. proportionality of potentials with point support, are needed to obtain the Martin compactification of W.

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